Sample ISI problems from Ch. 8 of Proakis 2002– ECE 588

(Revised and updated in line with the content of ECE 588)

Q1 (8.10 of Proakis) : A channel response is

$$C(f)=\begin{cases}1 & |f|\le W \\ 0 & |f|>W\end{cases} \tag{1.1}$$

If $W=100$ kHz, assuming we employ ASK at $M=2$ and we are free to adapt transmitter and receiver filter responses, i.e. $G_T(f)$ and $G_R(f)$, determine the highest bit rate that can be achieved through this channel. Find the required ε_b/N_0, i.e. the signal to noise ratio SNR to achieve a probability error $P_e=10^{-7}$. Also find the power at the highest bit rate available, if $N_0=4.1\times10^{-21}$ W/Hz.

Solution to Q1 : From Notes on ISI_Sept 2012_HTE, we know that if T is our symbol (in the case bit duration as well), then, for ISI free transmission, we must have $R=1/T\le 2W$, where the inverse of T is set to the symbol (in this case bit as well) rate. Thus $R_{\max}=1/T=2W=200$ kbps. We know from the first two terms of (6.30) of Notes on Dimensionality of Signal_Sept 2012_HTE, probability of error $M=2$ is

$$P_e=Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)=Q\left(\sqrt{2\text{SNR}}\right) \tag{1.2}$$

where $Q(\ )$ is the complimentary error which can well be approximated by

$$Q(x)\simeq\frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \qquad \text{for } x\ge 3 \tag{1.3}$$

Using (1.2) and (1.3), for the given $P_e=10^{-7}$, we get

$$Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)=Q\left(\sqrt{2\text{SNR}}\right)\simeq\frac{1}{\sqrt{4\pi\ \text{SNR}}}\exp(-\text{SNR}) \tag{1.4}$$

Finding SNR from (1.4) is a nonlinear operation, which means we need to plot the expression to get the zero crossing. This way, we get

$$\text{SNR}\simeq 13.52\simeq 11.3\text{ dB} \tag{1.5}$$

To go to the power, we simply use the expression that

$$P_b=\frac{\varepsilon_b}{T}=\frac{N_0}{T}\text{SNR}\simeq 1.1\times10^{-14}\text{ W}\simeq -109.5\text{ dBm} \tag{1.6}$$

Q2 (8.11 of Proakis) : Show that if a pulse has a raised cosine characteristics as shown below

$$x_{rc}(t) = \frac{\cos(\pi\alpha t/T)}{1-4\alpha^2 t^2 T^2}\,\mathrm{sinc}\left(\frac{t}{T}\right) \tag{2.1}$$

then, the zero ISI condition, i.e.

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{2.2}$$

is indeed satisfied.

Solution to Q2 : (Pasted directly from Proakis 2002)

The pulse $x(t)$ having the raised cosine spectrum is

$$x(t) = \mathrm{sinc}(t/T)\frac{\cos(\pi\alpha t/T)}{1-4\alpha^2 t^2/T^2}$$

The function $\mathrm{sinc}(t/T)$ is 1 when $t = 0$ and 0 when $t = nT$. On the other hand

$$g(t) = \frac{\cos(\pi\alpha t/T)}{1-4\alpha^2 t^2/T^2} = \begin{cases} 1 & t = 0 \\ \text{bounded} & t \neq 0 \end{cases}$$

The function $g(t)$ needs to be checked only for those values of t such that $4\alpha^2 t^2/T^2 = 1$ or $\alpha t = \frac{T}{2}$. However,

$$\lim_{\alpha t \to \frac{T}{2}} \frac{\cos(\pi\alpha t/T)}{1-4\alpha^2 t^2/T^2} = \lim_{x \to 1} \frac{\cos(\frac{\pi}{2}x)}{1-x}$$

and by using L'Hospital's rule

$$\lim_{x \to 1} \frac{\cos(\frac{\pi}{2}x)}{1-x} = \lim_{x \to 1} \frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right) = \frac{\pi}{2} < \infty$$

Hence,

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

meaning that the pulse $x(t)$ satisfies the Nyquist criterion.

Q3 (8.15 of Proakis) : A telephone channel has a bandpass characteristics such that

$$C(f) = \begin{cases} 1 & |f| \leq 3000 \text{ Hz} \\ 0 & \text{elsewhere} \end{cases} \tag{3.1}$$

Select a symbol rate to achieve an ISI free bit rate of $R_b = 14400$ bps signal transmission through this channel.

Solution to Q3 : Here, the (one sided) bandwidth is $W = 3000$ Hz , therefore the maximum achievable (symbol) rate over this channel is (if $x(t)$ is chosen as a sinc pulse)

$$R_s = \frac{1}{T} = 2W = 6000 \text{ symbols / s} \tag{3.2}$$

Now we attempt to use an M ary ASK, with $M = 2^k$, where k is the number of binary waveforms we group into a symbol, then

$$\frac{R_b}{R_s} = 2.4, \qquad k = \text{next integer to } (2.4) = 3 \tag{3.3}$$

Under these circumstances, we can revise R_s to $R_s = R_b / k = 4800 \text{ symbols / s}$. Then we can envisage the use of raised cosine response (instead of sinc) with

$$X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \dfrac{1-\alpha}{2T} \\ \dfrac{T}{2} \left\{1+\cos\left[\dfrac{\pi T}{\alpha}\left(|f| - \dfrac{1-\alpha}{2T}\right)\right]\right\} & \dfrac{1-\alpha}{2T} \le |f| \le \dfrac{1+\alpha}{2T} \\ 0 & |f| > \dfrac{1+\alpha}{2T} \end{cases} \tag{3.4}$$

$$x_{rc}(t) = \frac{\cos(\pi\alpha t / T)}{1 - 4\alpha^2 t^2 / T^2} \text{sinc}\left(\frac{t}{T}\right) \tag{3.5}$$

We know that the point where $X_{rc}(f) = 0$ is the point where $f = W = 3000 \text{ Hz}$. Thus from the last line of (3.4), it is possible to estimate the rolloff factor α, as follows

$$W = \frac{1+\alpha}{2T}\ ,\ 3000 = \frac{1+\alpha}{2} \times 4800\ ,\ \alpha = 0.25 \tag{3.6}$$

Q4 (8.18 of Proakis) : A telephone channel passes frequencies in the band 300 Hz to 3300 Hz. A modem that operates at 9600 bps is to be operated over this channel. If QAM is to be used, find which M ary scheme is to be used, estimate the roll-off factor for the raised cosine spectrum that will be utilized for this purpose.

Solution to Q4 : Our bandwidth of $2W$ is

$$2W = 3300 - 300 = 3000 \text{ Hz} \tag{4.1}$$

Our symbol rate should, R satisfy the following (if raised cosine spectrum is to be employed)

$$R = \frac{1}{T} < 2W \tag{4.2}$$

where T is the symbol duration. We may choose 2400 symbols / s. To convert a bit rate of 9600 bps to 2400 symbols / s, we need to group the binary waveforms into symbols such that

$$k = \frac{9600}{2400} = 4, \qquad M = 2^k = 16 \tag{4.3}$$

Thus we need 16 QAM scheme to pass our signal through this telephone channel.

We can estimate the rolloff factor as follows

$$\frac{1+\alpha}{2T} = W, \qquad \alpha = 0.25 \tag{4.4}$$

Q5 (8.20 of Proakis) : A telephone channel passes frequencies in the band 600 Hz to 3000 Hz. Evaluate the followings

a) The design parameters if a 4 PSK is used to transmit a data rate of 2400 bps.
b) The design parameters if a 4 PSK is used to transmit a data rate of 4800 bps.

Solution to Q5 : Here the bandwidth is

$$2W = 3000 - 600 = 2400 \text{ Hz} \tag{5.1}$$

a) In 4 PSK, each symbol carries two binary waveform, then the data rate of 2400 bps becomes 1200 symbols / s, this way the symbol duration becomes $T = 1/W$, which means that we can use a raised cosine spectrum with a rolloff factor of $\alpha = 1$. Then the raised cosine spectrum will be

$$X_{rc}(f) = \frac{T}{2}\left[1+\cos\left(\pi T|f|\right)\right] = \frac{1}{2400}\left[1+\cos\left(\frac{\pi}{1200}|f|\right)\right] = \frac{1}{2400}\cos^2\left(\frac{\pi}{2400}|f|\right) \tag{5.2}$$

b) If the data rate is raised to 4800 bps, then symbol rate in 4 PSK will become 2400 symbols / s, this way the symbol duration becomes $T = 1/2W$, which means that we now have to use a rectangular spectrum, which in turn corresponds to sinc pulse along time axis. Note that in this case the rolloff factor is $\alpha = 0$.

$$X(f) = \begin{cases} T & |f| \le W \text{ Hz} \\ 0 & \text{elsewhere} \end{cases} \quad , \quad x(t) = \text{sinc}\left(\frac{t}{T}\right) \tag{5.3}$$

Q6 (8.21 of Proakis) : A 4 KHz bandpass channel is to be used to transmit a bit rate of 9600 bps. Design an appropriate QAM scheme, determine the average power to achieve a bit error rate of 10^{-6}. Use raised cosine spectrum with a rolloff factor of $\alpha = 0.5$ and take noise spectral density $N_0 / 2 = 10^{-10}$ W/Hz.

Solution to Q6 : Since the bandwidth of the channel is $2W = 4$ kHz, we cannot exceed a maximum symbol rate of $R = 2W = 4$ ksymbols / s. But it is required that we utilize $\alpha = 0.5$, then

$$R(1+\alpha) = 2W, \qquad R \simeq 2667 \text{ symbols / s} \tag{6.1}$$

Since

$$4 > \frac{9600}{2667} > 3 \tag{6.2}$$

we have to choose $k = 4$, $M = 2^k = 16$. Thus 16 QAM is required to pass a 9600 bps signal through this bandlimited channel.

Q7 (8.28 of Proakis) : To increase the channel throughput (i.e. symbol rate), an ASK signal with a half symbol duration

$$v(t) = \sum_{k=-\infty}^{\infty} a_k g_T(t - kT/2) \tag{6.1}$$

is transmitted over an ideal channel, i.e.

$$C(f) = \begin{cases} 1 & |f| \le W \\ 0 & |f| > W \end{cases} \tag{6.2}$$

Determine the ISI values at the output of matched filter demodulator if a raised cosine filter with a rolloff factor of unity is used.

Solution to Q7 : The output of the matched filter demodulator is

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} g_T(\tau - kT/2)\, g_R(t-\tau)\, d\tau + \eta(t) \\ &= \sum_{k=-\infty}^{\infty} a_k x(t - kT/2) + \eta(t) \end{aligned} \tag{6.3}$$

where $x(t)$ is the raised cosine response with unity rolloff factor and symbol duration of T, thus

$$x(t) = \frac{\cos(\pi t/T)}{1 - 4t^2/T^2} \operatorname{sinc}\left(\frac{t}{T}\right) \tag{6.4}$$

Upon inserting (6.4) into (6.3) and sampling at integer values of $t = mT/2$, we get

$$y(mT/2) = \frac{a_{m-1}}{2} + a_m + \frac{a_{m+1}}{2} + \eta(mT/2) \tag{6.5}$$

It is important to realize that in (6.5), a_m is the desired symbol to be detected, while a_{m-1} and a_{m+1} represent the ISI symbols. The contribution from other prior and later symbols becomes zero due to zero crossings of (6.4).

Exercise 1 : Solve Q7 and find the ISI terms for the case of symbol duration being reduced to a quarter of the original symbol duration, or symbol rate being quadruple.

Q8 (8.42 of Proakis) : A channel has the following response.

$$C(f) = 1 + 0.3\cos(2\pi f T) \tag{8.1}$$

Determine the frequency responses of transmitter and receiver filters, i.e. $G_T(f)$ and $G_R(f)$ if a data rate of $1/T$ is to be passed through this channel with zero ISI. Assume that $G_T(f)$ and $G_R(f)$ are matched and each share the square root of the channel response. Take the rolloff factor to be $\alpha = 0.5$.

Solution to Q8 : For a rolloff factor of $\alpha = 0.5$, from (3.13) of Notes on ISI_Sept 2012_HTE, we get

$$X_{rc}(f) = G_T(f)C(f)G_R(f) = \begin{cases} T & 0 \le |f| \le \frac{1}{4T} \\ \frac{T}{2}\left\{1+\cos\left[2\pi T\left(|f| - \frac{1}{4T}\right)\right]\right\} & \frac{1}{4T} \le |f| \le \frac{3}{4T} \\ 0 & |f| > \frac{3}{4T} \end{cases} \tag{8.2}$$

If we are to match $G_R(f)$ to $G_T(f)$, then

$$|G_T(f)| = \sqrt{\frac{|X_{rc}(f)|}{|C(f)|}} \;,\; |G_R(f)| = \sqrt{\frac{|X_{rc}(f)|}{|C(f)|}} \;,\; |G_T(f)| = |G_R(f)| \tag{8.3}$$

Using (8.1) and (8.2), (8.3) will become

$$|G_T(f)| = |G_R(f)| = \begin{cases} \sqrt{\frac{T}{1+0.3\cos(2\pi f T)}} & 0 \le |f| \le \frac{1}{4T} \\ \sqrt{\frac{T\left[1++\cos\left[2\pi T\left(|f| - 1/4T\right)\right]\right]}{2\left[1+0.3\cos(2\pi f T)\right]}} & \frac{1}{4T} \le |f| \le \frac{3}{4T} \\ 0 & |f| > \frac{3}{4T} \end{cases} \tag{8.4}$$

Exercise 2 : Find the symbol rate in Q8 for the solution given above. Solve Q8 by taking a rolloff factors of $\alpha = 0$ and $\alpha = 1$. Determine the maximum symbol rate for each case that can be obtained without ISI. Plot in Matlab, the graphs of $C(f)$, $G_T(f)$ and $G_R(f)$.

Q9 (8.43 of Proakis) : A channel has the following response.

$$C(f)=\begin{cases}\dfrac{1}{1+j\dfrac{f}{2400}} & |f|\le 2400 \text{ Hz} \\ 0 & |f|>2400 \text{ Hz}\end{cases} \tag{9.1}$$

A 4 ASK signal having a bit rate of 9600 bps is transmitted over this channel. Find the frequency response of the optimum transmitter and receiver filters.

Solution to Q9 : At $M=4=2^k$, $k=2$, hence a bit rate of $R_b=9600$ bps is converted into a symbol rate of $R_s=R_b/2=4800$ symbols/sec, thus the inverse symbol duration is equal to 2×2400 Hz, that is $1/T=R_s=2W$. From Notes on ISI_Sept 2012_HTE, we see that in such circumstances, we may a sinc pulse along time axis, which corresponds to a rectangular function along frequency axis, which can be expressed as

$$X(f)=\begin{cases}T=1/2W & |f|<W \\ 0 & \text{otherwise}\end{cases} \quad , \; W=2400 \text{ Hz} \; , \; x(t)=\text{sinc}\left(\frac{t}{T}\right) \tag{9.2}$$

In this case the inverse of the square of the channel frequency response can be shared between the transmitter and receiver filters as follows ;

$$|G_T(f)|=|G_R(f)|=\begin{cases}T^{0.5}\left[1+\left(\dfrac{f}{2400}\right)^2\right]^{0.25} & |f|<W \\ 0 & \text{otherwise}\end{cases} \tag{9.3}$$

Q10 (8.44 of Proakis) : An uequalized channel produces the following response

$$x_m=\begin{cases}0.3 & m=1 \\ 0.9 & m=0 \\ 0.3 & m=-1 \\ 0 & \text{otherwise}\end{cases} \tag{10.1}$$

a) Design a three tap equalizer such that

$$q_m=\begin{cases}1 & m=0 \\ 0 & m=\pm 1\end{cases} \tag{10.2}$$

b) Find q_m for $m=\pm 2, \; \pm 3$

Solution to Q10 : Since the equalizing action is going to be based on FIR devices, then (10.1) means that

$$x(t)=\sum_{m=-1}^{1} x_m\delta(t-mT)=x_{-1}\delta(t+T)+x_0\delta(t)+x_1\delta(t-T)$$
$$=0.3\delta(t+T)+0.9\delta(t)+0.3\delta(t-T)$$
$$q_m=\sum_{n=-1}^{1} c_n x_{m-n} \quad \text{with} \quad m=-1,\ 0,\ 1 \quad n=-1,\ 0,\ 1 \tag{10.3}$$

To solve the three tap coefficients of the equalizer, from (10.3), we construct the following matrix equality

	$m=-1$	$m=0$	$m=1$
$n=-1$	0.9	0.3	0
$n=0$	0.3	0.9	0.3
$n=1$	0	0.3	0.9

$\Leftarrow$ **X** matrix constructed from x_{m-n} with $m=-1,\ 0,\ 1 \ \ n=-1,\ 0,\ 1$

$$\mathbf{q}=\begin{pmatrix}0\\1\\0\end{pmatrix}=\mathbf{Xc}=\begin{pmatrix}0.9 & 0.3 & 0\\0.3 & 0.9 & 0.3\\0 & 0.3 & 0.9\end{pmatrix}\begin{pmatrix}c_{-1}\\c_0\\c_1\end{pmatrix} \tag{10.4}$$

Solving the lower line of (10.4), we get

$$\begin{pmatrix}c_{-1}\\c_0\\c_1\end{pmatrix}=\begin{pmatrix}-0.4762\\1.4286\\-0.4762\end{pmatrix} \tag{10.5}$$

Exercise 3 : Solve part b) of Q10.

Exercise 4 : Solve problem 8.45 of Proakis 2002.