Transmission Through Bandlimited AWGN Channels

HTE - 24.09.2012

1. Definition of a Bandlimited Channel

Previously we assumed that the communication channel was unlimited, i.e. all pass channel expressed as

$$C(f) = 1 \ , \ c(t) = \delta(t) \ , \ \delta(t) \ : \text{Time delta function} \tag{1.1}$$

Fig. 1.1 illustrates $C(f)$ and $c(t)$ of an unlimited channel as defined in (1.1). In practice the frequency response $C(f)$ and the impulse response $c(t)$ would hardly be like (1.1). More realistic plots are given in Fig. 1.2.

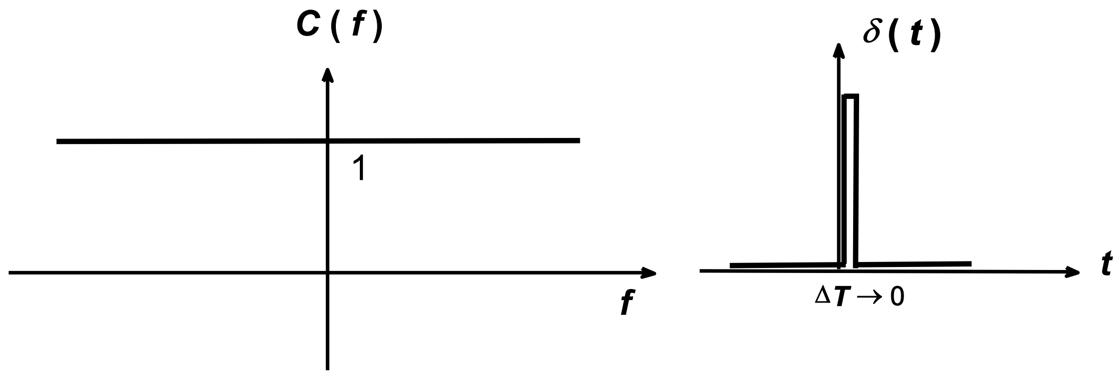
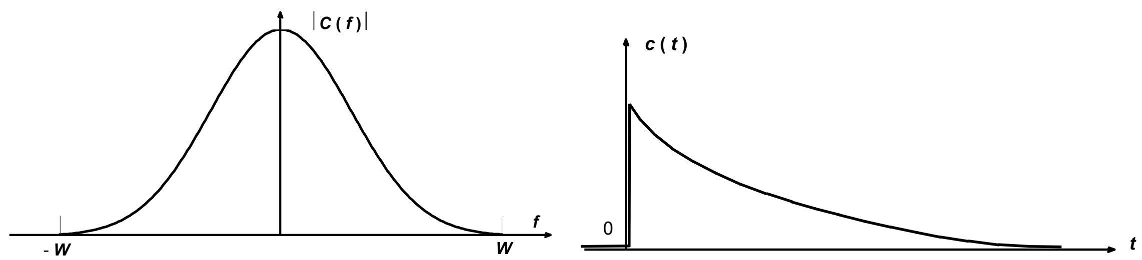


Fig. 1.1 Frequency and time responses of an unlimited channel.



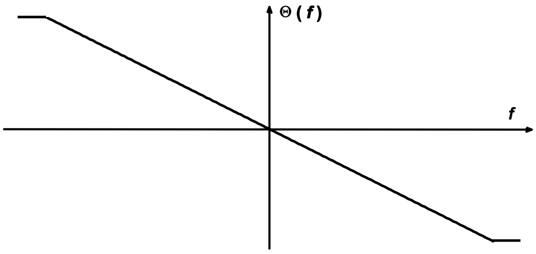


Fig. 1.2 Frequency, time and phase responses of a typical bandlimited (bandpass) channel.

As seen from Fig. 1.2, the frequency response of a the bandlimited channel, $C(f)$ is zero when $|f|>W$. And $C(f)$ acts like a low pass filter within $|f|\leq W$. The same can also be detected from $c(t)$ of Fig. 1.2.

Assume that a transmitter send a time signal of $g_T(t)$, to a communication channel whose time and frequency responses are $C(f)$ and $c(t)$, at the output of the channel we will obtain in time domain

$$h(t)=\int_{-\infty}^{\infty} c(\tau)\, g_T(t-\tau)\, d\tau = c(t) * g_T(t) \tag{1.2}$$

Alternatively, in frequency domain, the output will be

$$H(f)=C(f)G_T(f) \tag{1.3}$$

where $H(f)$ and $G_T(f)$ are the Fourier transforms of $h(t)$ and $g_T(t)$. At receiver we pass the received signal plus noise, i.e. $h(t)+n(t)$ though a matched filter (MF) whose frequency response is

$$G_R(f)=H^*(f)\exp(-2j\pi f T_s) \tag{1.4}$$

T_s indicates the sampling instance. So after sampling the output of the matched filter at $t=T_s$

$$y_s(t=T_s)=\int_{-\infty}^{\infty} |H(f)|^2\, df = \varepsilon_h \tag{1.5}$$

where ε_h is the energy in the channel output signal $h(t)$. Noise has a spectral density of $S_n(f)=N_0/2$ at input to MF, which means at the output of MF, the spectral density will be

$$S_0(f)=|H(f)|^2\, S_n(f)=|H(f)|^2\, N_0/2 \tag{1.6}$$

So the noise power can be obtained by integrating (1.6) over the whole frequency spectrum (actually this will be limited to the bandwidth of the matched filter)

$$P_n = \int_{-\infty}^{\infty} S_0(f)\, df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df = \frac{\varepsilon_h N_0}{2} \tag{1.7}$$

By setting the signal power to the square of (1.5), i.e. $P_s = \varepsilon_h^2$, we can find the signal to noise ratio $(\mathrm{SNR})_o$ at the output of the matched filter as

$$(\mathrm{SNR})_o = \left(\frac{S}{N}\right) = \frac{P_s}{P_n} = \frac{\varepsilon_h^2}{\varepsilon_h N_0 / 2} = \frac{2\varepsilon_h}{N_0} \tag{1.8}$$

(1.8) is the same as the obtained previously except ε_h was replaced by ε_s. In the present case ε_s would be the energy in $g_T(t)$ or $G_T(f)$. In view of (1.3) and $C(f) \le 1$ for all f, we deduce that $\varepsilon_h \le \varepsilon_s$. This means a fall in the signal to noise ratio for the case bandlimited channel. Furthermore (1.4) implies that the design of the matched filter can no longer be based on transmitted signal only, but it also requires the knowledge of channel characteristics.

Example 1.1 : A transmitted signal $g_T(t)$ is given by

$$g_T(t) = \begin{cases} \frac{1}{2}\left\{1 + \cos\left[\frac{2\pi}{T}\left(t - \frac{T}{2}\right)\right]\right\} & , \quad 0 \le t \le T \\ 0 & \quad \text{elsewhere} \end{cases} \tag{1.9}$$

$g_T(t)$ passes through a channel whose response is as shown in Fig. 1.3. Find the ratio of $\varepsilon_h / \varepsilon_s$ against the product of signal duration and the channel's bandwidth.

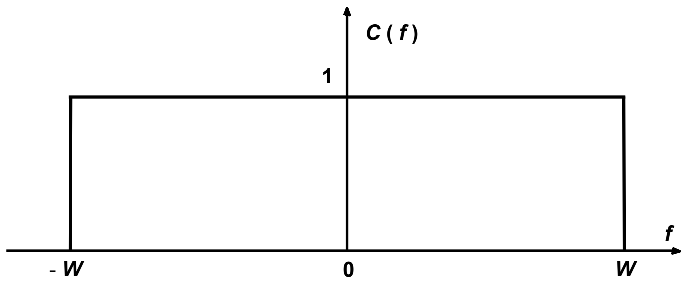
Solution : By the Fourier transform of (1.9), we get $G_T(f)$ as

$$G_T(f) = \frac{T}{2}\frac{\sin(\pi f T)}{\pi f T\left(1 - f^2T^2\right)}\exp(-j\pi f T) = \frac{T}{2}\frac{\mathrm{sinc}(fT)}{\left(1 - f^2T^2\right)}\exp(-j\pi f T) \tag{1.10}$$

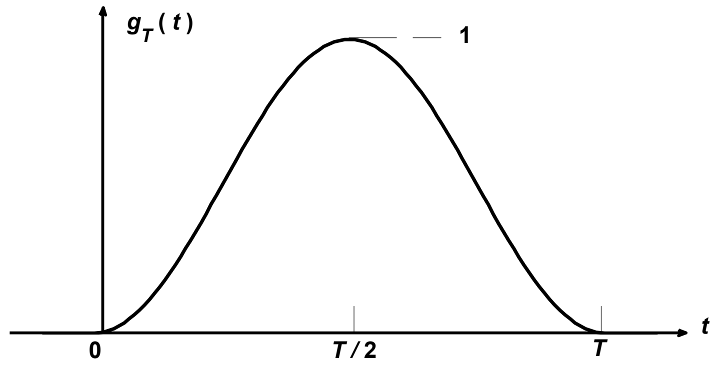
Using (1.3), we obtain the output of the channel as

$$\begin{aligned} H(f) &= C(f)G_T(f) \\ &= \begin{cases} G_T(f) & , \quad |f| \le W \\ 0 & \quad \text{otherwise} \end{cases} \end{aligned} \tag{1.11}$$

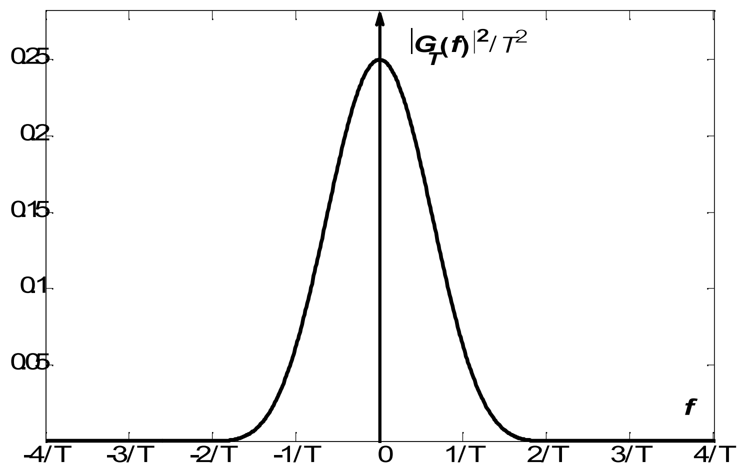
Channel response, $C(f)$, $g_T(t)$ and $|G_T(f)|^2 / T^2$ are plotted in Fig 1.3



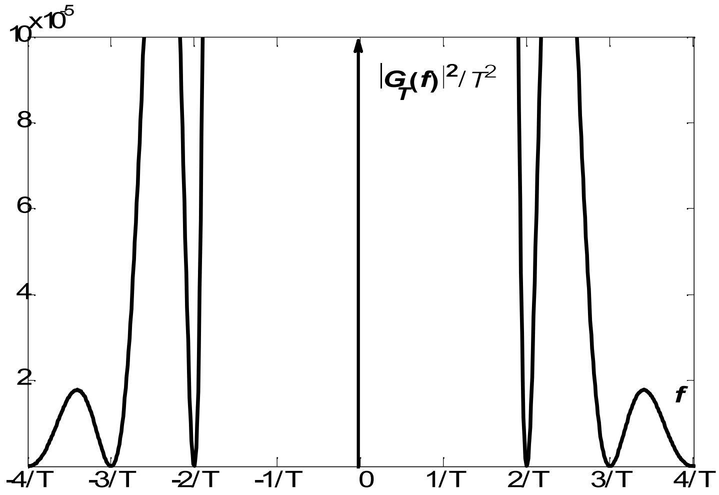
a) Channel response, $C(f)$



b) Transmitted signal, $g_T(t)$



c) The spectrum of transmitted signal , $\left|G_T(f)\right|^2 / T^2$



d) Exploded view of the spectrum of transmitted signal , $\left|G_T\left(f\right)\right|^2/T^2$

Fig. 1.3 The plots of $C\left(f\right)$, $g_T\left(t\right)$ and $\left|G_T\left(f\right)\right|^2/T^2$ in Example 1.1.

Note that due to side lobes of $\left|G_T\left(f\right)\right|^2/T^2$ being much smaller than the main lobe, we have had to shown an exploded view in Fig. 1.3d. Now the signal component at the output of the filter matched to $H\left(f\right)$ is

$$\varepsilon_h = \int_{-W}^{W} \left|G_T\left(f\right)\right|^2 df = \frac{1}{4\pi^2}\int_{-W}^{W} \frac{\sin^2\left(\pi fT\right)}{f^2\left(1-f^2T^2\right)^2} df = \frac{T}{4\pi^2}\int_{-WT}^{WT} \frac{\sin^2\left(\pi\alpha\right)}{\alpha^2\left(1-\alpha^2\right)^2} d\alpha \qquad (1.12)$$

(1.12) applies to the situation where the channel bandwidth is limited to W. In the case of unlimited channel however, (1.12) will be

$$\varepsilon_s = \int_{-\infty}^{\infty} \left|G_T\left(f\right)\right|^2 df = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{\sin^2\left(\pi fT\right)}{f^2\left(1-f^2T^2\right)^2} df = \frac{T}{4\pi^2}\int_{-\infty}^{\infty} \frac{\sin^2\left(\pi\alpha\right)}{\alpha^2\left(1-\alpha^2\right)^2} d\alpha \qquad (1.13)$$

By taking the ratio of $\varepsilon_h/\varepsilon_s$, it is possible to estimate how much of the transmitted signal energy will reach the receiver for a given bandwidth. From (1.12) and (1.13) we can write

$$\frac{\varepsilon_h}{\varepsilon_s} = \frac{\displaystyle\int_{-WT}^{WT} \frac{\sin^2\left(\pi\alpha\right)}{\alpha^2\left(1-\alpha^2\right)^2} d\alpha}{\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2\left(\pi\alpha\right)}{\alpha^2\left(1-\alpha^2\right)^2} d\alpha} \qquad (1.14)$$

As understood from (1.14), it more reasonable to examine the variation of $\varepsilon_h/\varepsilon_s$ against the dimensionless quantity WT rather than the absolute bandwidth W. This is done in Fig. 1.4.

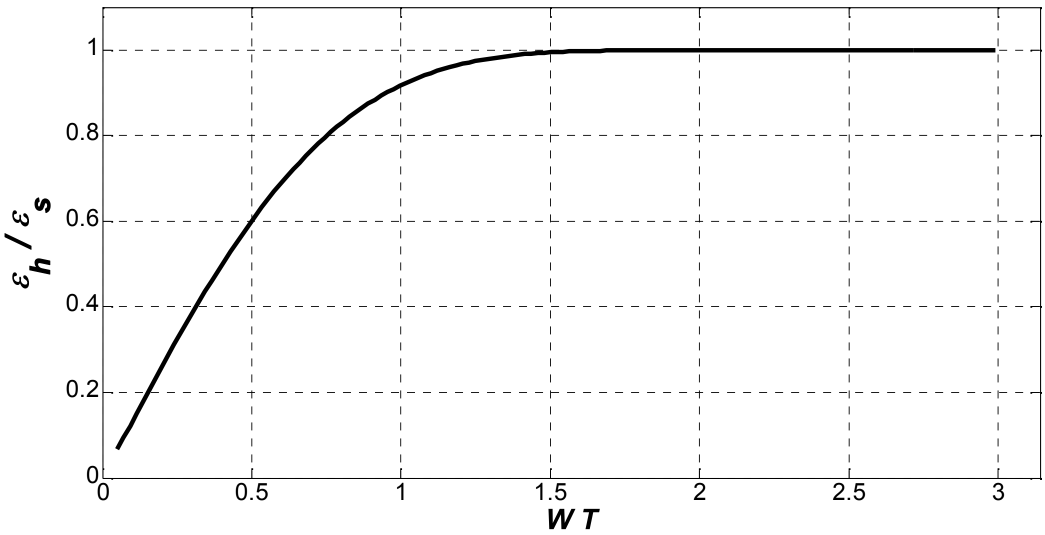


Fig. 1.4 The variation of the normalized energy in band limited channel against the product of bandwidth and symbol duration.

As can be seen from Fig. 1.4, with the given transmitted signal of (1.9) and the rectangular bandlimited channel response, it is possible to pass 100 % of the signal power nearly beyond the values of $WT \geq 2$. This is twice the bandwidth requirement for a symbol duration of T. Normally we expect to assign a bandwidth of $1/T$ to a symbol duration of T. In summary we need better designed signal shapes that the one given in (1.9).

It is important to appreciate that the above phenomena is purely due to bandlimiting action of the channel. There is no harm in modelling bandpass channels as low pass filters, since the shift in the central frequency has no effect on our analysis.

Fig. 1.5 shows the bandlimiting and ISI effects of a when rectangular $g_T(t)$ passes through unlimited and bandlimited channels respectively.

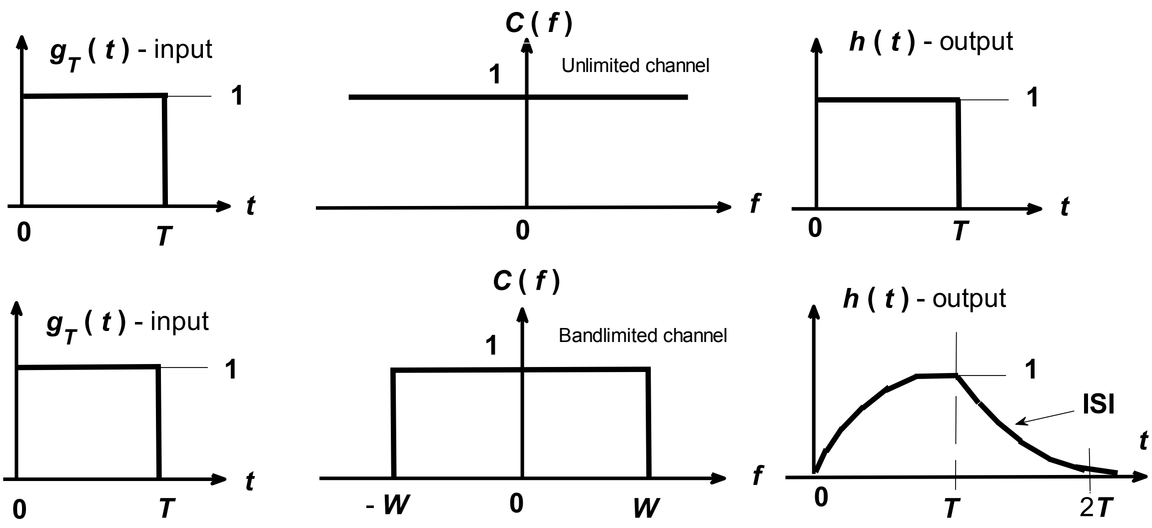


Fig. 1.5 The effect of bandlimited channel on a rectangular $g_T(t)$.

As seen from Fig. 1.5, if we try to pass a rectangular $g_T(t)$ through an unlimited channel, then the output $h(t)$ is an exact replica of the input to $g_T(t)$. Note that the time delay through the channel is omitted here. On the other hand, pass the same rectangular $g_T(t)$ through a bandlimited channel creates two effects

- Intersysmbol interference (ISI) occurs, which means that the falling edge of $g_T(t)$ extends into the next symbol interval of $T \leq t \leq 2T$, thereby making the correct detection of the symbol there more difficult.
- Due to ISI, part of the energy of the symbol in the interval $0 \leq t \leq T$ is lost, again causing difficulties in the correct detection of the symbol in the interval of $0 \leq t \leq T$.

Exercise 1. 1: Convert $g_T(t)$ in (1.9) into a rectangular pulse of

$$g_T(t) = \begin{cases} 1 & , \quad 0 \leq t \leq T \\ 0 & \quad \text{elsewhere} \end{cases} \tag{1.15}$$

Calculate and plot $\varepsilon_h / \varepsilon_s$ for the rectangular $g_T(t)$ of (1.15). Compare your results with those of Example 1.1. Note that for this exercise, you can benefit from the m code ISIExp1.m on the course web page. This m code illustrates what happens to a rectangular transmitted signal, i.e. the case of (1.15) if the channel is unlimited, bandlimited (quite wide wide, wide and narrow) and RC low pass.

Exercise 1. 2: From Proakis text book and solution manual, after studying the problems and the solutions of 8.2 and 8.9, design ISIExp1.m, so that the channel is a low pass filter as defined in problem 8.9 of Proakis text book, then run ISIExp1.m to see if you get the same results as given in the solution of problem 8.9.

2. Transmission of Binary ASK Signals Through Bandlimited Channels

The block diagram of the ASK system in question is given in Fig. 2.1.

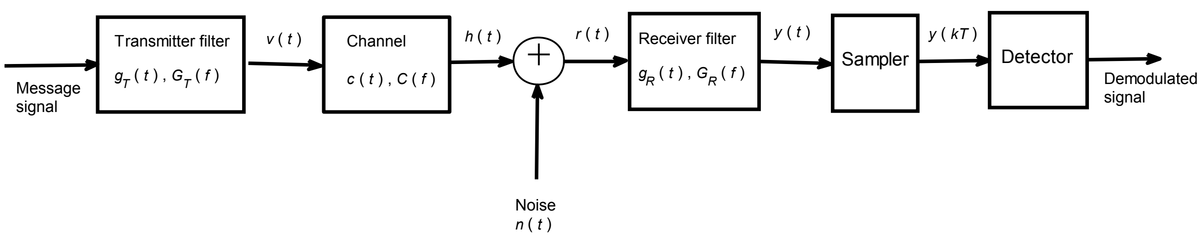


Fig. 2.1 Block diagram of the analysed communication system

We assume that the signal at the output of the transmitter filter is

$$v(t)=\sum_{n=-\infty}^{\infty} a_n g_T(t-nT) \tag{2.1}$$

Where, T is the symbol duration, $\{a_n\}$ contain the sequence of amplitude levels assigned to the M ary ASK in question. For instance for $M=8$, $\{a_n\}$ might simply be $\{a_n\}=\{\pm 1,\ \pm 3,\ \pm 5,\ \pm 7\}$. $g_T(t)$ is the impulse response of the transmitting filter or the shaping waveform furnished by the transmitter filter to the ASK signal to be transmitted. Note that during the analysis of unlimited channels, it was sufficient to consider one (any) symbol interval to arrive at results. Here because of the presence of bandlimiting, symbols will smear out to neighbouring time intervals. Therefore we take the message signal in the format of (2.1). At the output of the channel, we will have

$$r(t)=\sum_{n=-\infty}^{\infty} a_n h(t-nT)+n(t) \tag{2.2}$$

where $h(t)=c(t)*g_T(t)$ represents the cascaded impulse response of the transmitter filter and the channel. $n(t)$ is additive white Gaussian noise (AWGN). The received signal of (2.2) will pass through a receiver filter whose time response $g_R(t)$ is matched to $h(t)$, thus the output of the receiver filter will be

$$y(t)=\sum_{n=-\infty}^{\infty} a_n x(t-nT)+\eta(t) \tag{2.3}$$

where

$$x(t)=h(t)*g_R(t)=g_T(t)*c(t)*g_R(t)\ \ ,\ \ \eta(t)=n(t)*g_R(t) \tag{2.4}$$

Hence and combined impulse response of the transmitter filter, the channel and the receiver filter, while $\eta(t)$ is the AWGN at output of the receiver filter. After sampling at $t=mT$ instance, we have

$$\begin{aligned}
y(mT) &= \sum_{n=-\infty}^{\infty} a_n x(mT-nT)+\eta(mT) \\
y_m &= \sum_{n=-\infty}^{\infty} a_n x_{m-n}+\eta_m\ \ ,\ \ y_m=y(mT)\ \ ,\ \ x_{m-n}=x(mT-nT)\ \ ,\ \ \eta_m=\eta(mT) \\
&= x_0 a_m+\sum_{n\neq m} a_n x_{m-n}+\eta_m\ \ ,\ m=0,\ \pm 1,\ \pm 2\cdots
\end{aligned} \tag{2.5}$$

In (2.5), the second line is just a short hand notation of the first line, while on the last line, we have singled out the symbol $n=m$ as the first term since that is the symbol we wish to recover at the sampling instance of $t=mT$. The middle term on the last line of (2.5) is the undesirable effect of other symbols at the chosen sampling instance and causes what is called intersymbol interference (ISI). The last term here represents noise. As seen from the term on the last line of (2.5), the transmitted symbol a_m is scaled by the parameter x_0 as a result of passing through the channel and receiver filter (matched to channel output, $h(t)$), where from (1.3) – (1.5) and (2.4) we can deduce x_0 to be

$$x_0 = \int_{-\infty}^{\infty} h^2(t)\, dt = \int_{-\infty}^{\infty} |H(f)|^2\, df = \int_{-W}^{W} |G_T(f)|^2 |C(f)|^2\, df = \varepsilon_h \tag{2.6}$$

By appropriately designing transmitter and receiver filters in Fig 2.1, it is possible to eliminate the ISI term, i.e. the middle term on the last line of (2.5). This we shall do next.

3. Signal Design for Zero ISI

We rewrite (2.5) as follows

$$y_m = x(0)a_m + \sum_{\substack{n=-\infty \\ n\neq m}}^{\infty} a_n x(mT - nT) \; + \eta(mT) \tag{3.1}$$

To eliminate ISI, the necessary and sufficient condition is that $x(mT - nT) = 0$ for $n \neq m$ and $x(0) \neq 0$. Assuming that we set $x(0) = 1$, then we can reduce this condition to

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{3.2}$$

According to Nyquist theorem, for the condition in (3.2) to be satisfied, it is necessary that

$$\sum_{n=-\infty}^{\infty} X\left(f + \frac{n}{T}\right) = T \tag{3.3}$$

Below, we give the proof that if (3.3) is valid, then (3.2) will be satisfied.

Initially we take the formal definition of inverse Fourier transform for $x(t)$ which is

$$x(t) = \int_{-\infty}^{\infty} X(f)\, \exp(j2\pi f t)\, df \tag{3.4}$$

At the sampling instance of $t = nT$, (3.4) will become

$$x(nT) = \int_{-\infty}^{\infty} X(f)\, \exp(j2\pi f nT)\, df \tag{3.5}$$

The infinite integral in (3.5) can be broken into $1/T$ (frequency) intervals such that

$$x(nT)=\sum_{n=-\infty}^{\infty}\int_{-1/2T}^{1/2T}X\left(f+\frac{n}{T}\right)\exp(j2\pi fnT)\,df$$

$$=\int_{-1/2T}^{1/2T}Z(f)\exp(j2\pi fnT)\,df \quad , \quad Z(f)=\sum_{n=-\infty}^{\infty}X\left(f+\frac{n}{T}\right) \tag{3.6}$$

As seen, the function $Z(f)$ is a periodic with a period of $1/T$. Hence it will have a Fourier series in the form of

$$Z(f)=\sum_{n=-\infty}^{\infty}z_n\exp(j2\pi fnT) \quad , \quad z_n=T\int_{-1/2T}^{1/2T}Z(f)\exp(-j2\pi fnT)\,df \tag{3.7}$$

Comparing (3.7) with (3.6), we get

$$z_n=Tx(-nT) \tag{3.8}$$

Now the necessary condition of (3.2) for zero ISI will be satisfied if in (3.9)

$$z_n=\begin{cases}T & n=0\\ 0 & n\neq 0\end{cases} \tag{3.9}$$

Substituting this into the summation (i.e. the first expression) of (3.7) and using the second expression of (3.6), we get

$$Z(f)=T=\sum_{n=-\infty}^{\infty}X\left(f+\frac{n}{T}\right) \tag{3.10}$$

This completes our proof. It is instructive to state that the condition in (3.10) essentially means that the infinite sum of spectrums of $X(f)$ at shifted periodic frequency intervals of n/T should be equal to the symbol duration T (constant).

Now we take a channel like the one given in Fig. 1.2, that is $C(f)=0$, when $|f|>W$. Then $X(f)=G_T(f)C(f)G_R(f)$ will extend up to $|f|=W$, but will be zero for $|f|>W$. In such circumstances, it is possible to identify three types $Z(f)$ depending on the relations between T and $2W$.

1) $1/T>2W$ or $(1/T)/(2W)>1$. In this case, the ends of the shifted periodic spectrums of $X(f)$ cannot touch each other, thus there is no possibility of satisfying $Z(f)=T$. This situation corresponds to symbol duration being too short, so essentially more channel bandwidth than the one provided is required to achieve a zero ISI condition. The other interpretation is that the signal bandwidth is larger than the bandwidth offered by the channel (see Figs. 3.1a).

2) $1/T = 2W$ or $(1/T)/(2W) = 1$. In this case, the ends of the shifted periodic spectrums of $X(f)$ begin to touch each other, but they have to be rectangular in shape so that $Z(f) = T$ is satisfied. Thus

$$X(f) = \begin{cases} T & |f| < W \\ 0 & \text{otherwise} \end{cases} \quad (3.11)$$

Taking the inverse Fourier transform of $X(f)$, we get

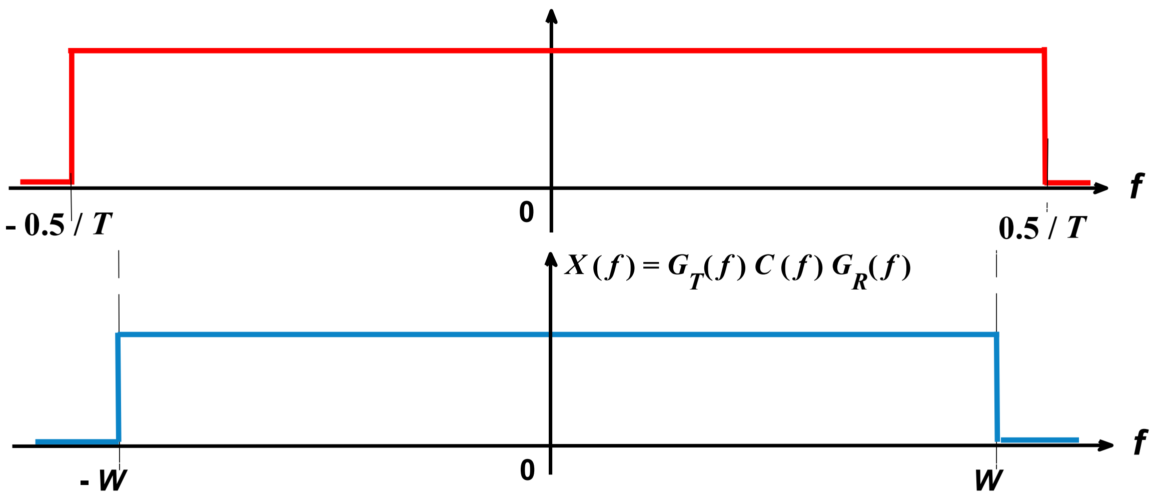
$$x(t) = \text{sinc}\left(\frac{t}{T}\right) \quad (3.12)$$

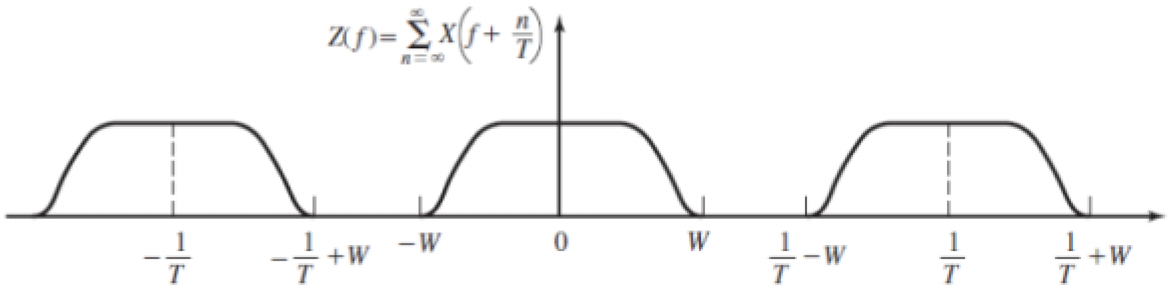
Although the zero ISI condition is satisfied by the sinc function given in (3.12), some disadvantages exist such that the tails of a sinc function does not converge rapidly.

3) $1/T < 2W$ or $(1/T)/(2W) < 1$. In this case, the shifted periodic spectrums of $X(f)$ overlap, so there are many choices of $X(f)$, consequently $x(t)$. Note that $1/T < 2W$ means that our signal can safely be accommodated within the channel bandwidth.

Plots of $Z(f)$ for the above examined cases are given in Fig. 3.1 (copied from Proakis 2002). Additionally our own plots, where the different ratios of $(1/T)/(2W)$ are shown, are also added as the first illustrations in Fig. 3.1.

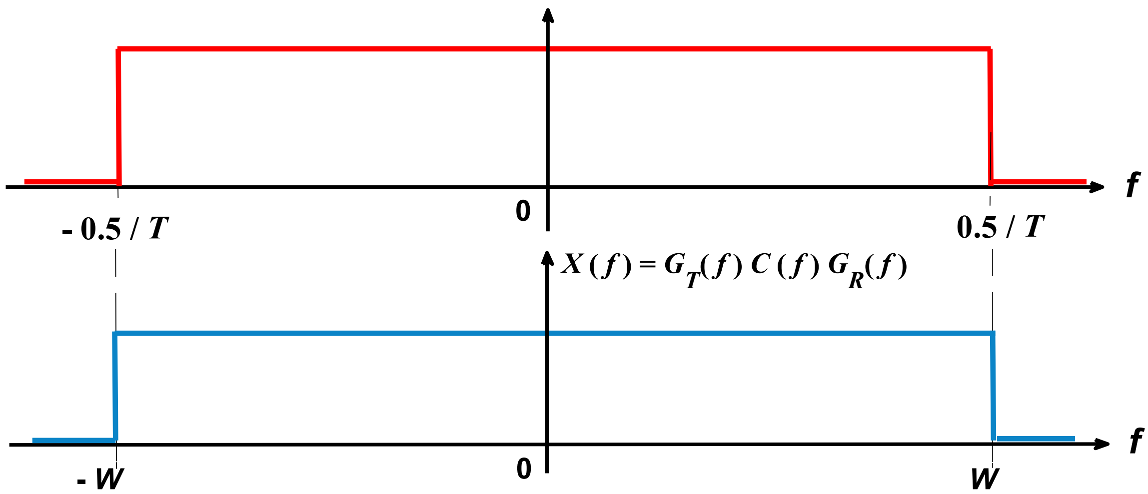
Case of signal bandwidth being larger than bandwidth of the bandpass channel , (1 / T) / (2 W) > 1

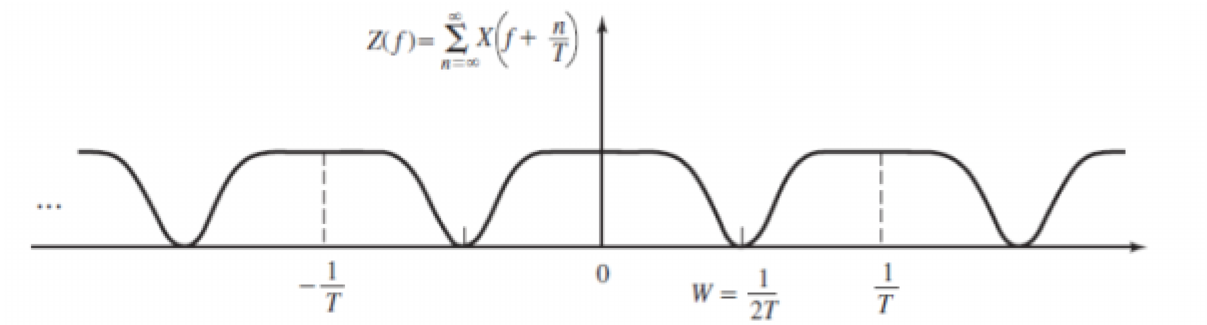




a) Plot of $Z(f)$ for $1/T > 2W$ or $(1/T)/(2W) > 1$

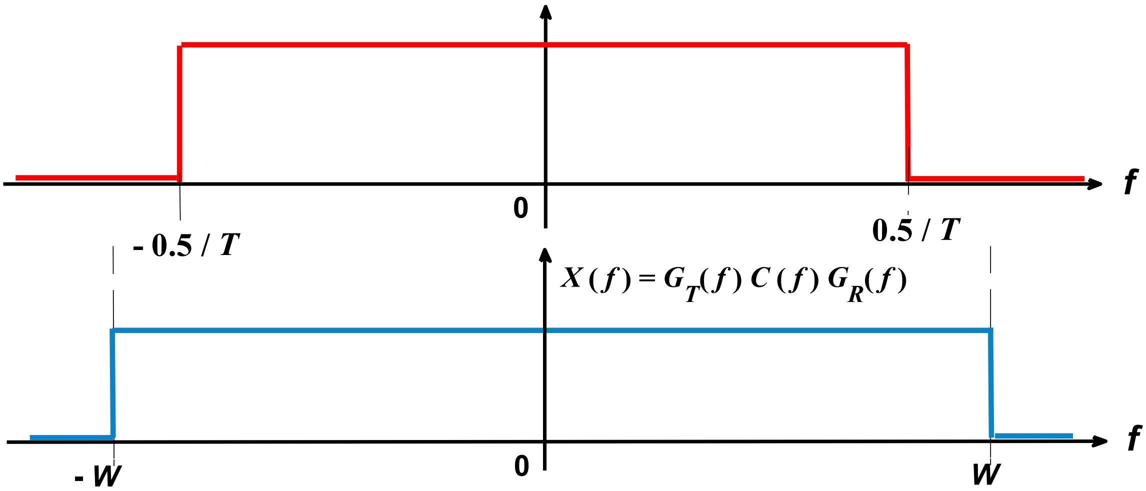
Case of signal bandwidth being equal to bandwidth of the bandpass channel , (1 / *T*) / (2 *W*) = 1

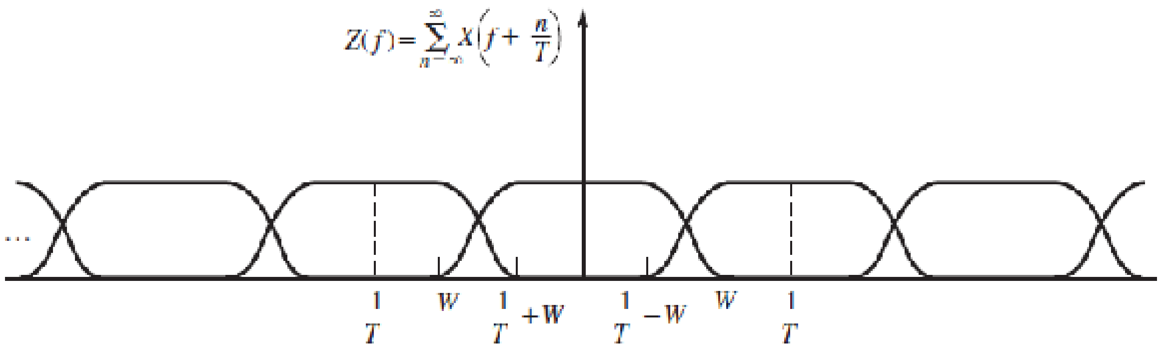




b) Plot of $Z(f)$ for $1/T = 2W$ or $(1/T)/(2W) = 1$

Case of signal bandwidth being smaller than bandwidth of the bandpass channel , (1 / T) / (2 W) < 1





c) Plot of $Z(f)$ for $1/T < 2W$ or $(1/T)/(2W) < 1$

Fig. 3.1 Plots of $1/T, 2W$ and $Z(f)$ for different ratios of $(1/T)/(2W)$.

A particularly popular choice for $X(f)$ is the raised cosine spectrum denoted as $X_{rc}(f)$ and given by

$$X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \dfrac{1-\alpha}{2T} \\ \dfrac{T}{2}\left\{1+\cos\left[\dfrac{\pi T}{\alpha}\left(|f| - \dfrac{1-\alpha}{2T}\right)\right]\right\} & \dfrac{1-\alpha}{2T} \le |f| \le \dfrac{1+\alpha}{2T} \\ 0 & |f| > \dfrac{1+\alpha}{2T} \end{cases} \tag{3.13}$$

where α is the rolloff factor and ranges in $0 \le \alpha \le 1$. It is important to realize that on the last line of (3.13), the reason for $X_{rc}(f)$ becoming zero is due to channel response being zero outside of $|W|$. Thus effectively for an arbitrary value of α in the range of zero to unity, we have $(1+\alpha)/2T = W$.

$x_{rc}(t)$ of (3.13) will be

$$x_{rc}(t) = \frac{\cos(\pi\alpha t/T)}{1-4\alpha^2 t^2/T^2}\,\mathrm{sinc}\left(\frac{t}{T}\right) \quad (3.14)$$

The plots of $X_{rc}(f)$ and $x_{rc}(t)$ for three different values of $\alpha = 0,\ 0.5,\ 1$ are given in Fig. 3.2. It is worth noting that $\alpha = 0$ corresponds to the case of (highest signalling rate) $1/T = 2W$, thus here $x_{rc}(t)$ turns into a sinc function. But at $\alpha = 1$, signalling rate is lowered to $1/T = W$ as seen from Fig. 3.1c. Fig. 3. 2b reveals that decay in the tails of $x_{rc}(t)$ accelerates as α approaches unity, confirming that a raised cosine shape is a better confined waveform than a sinc pulse. It is important to realize that $X_{rc}(f)$ of (3.13) is normalized in the sense that its integration over the whole range of frequencies (also when confined to $|f| < W$) is equal to unity. Furthermore $x_{rc}(t)$ is also normalized for $t = 0$ such that

$$\int_{-\infty}^{\infty} X_{rc}(f)\,df = \int_{-W}^{W} X_{rc}(f)\,df = 1 \quad , \quad x_{rc}(t=0) = 1 \quad (3.15)$$

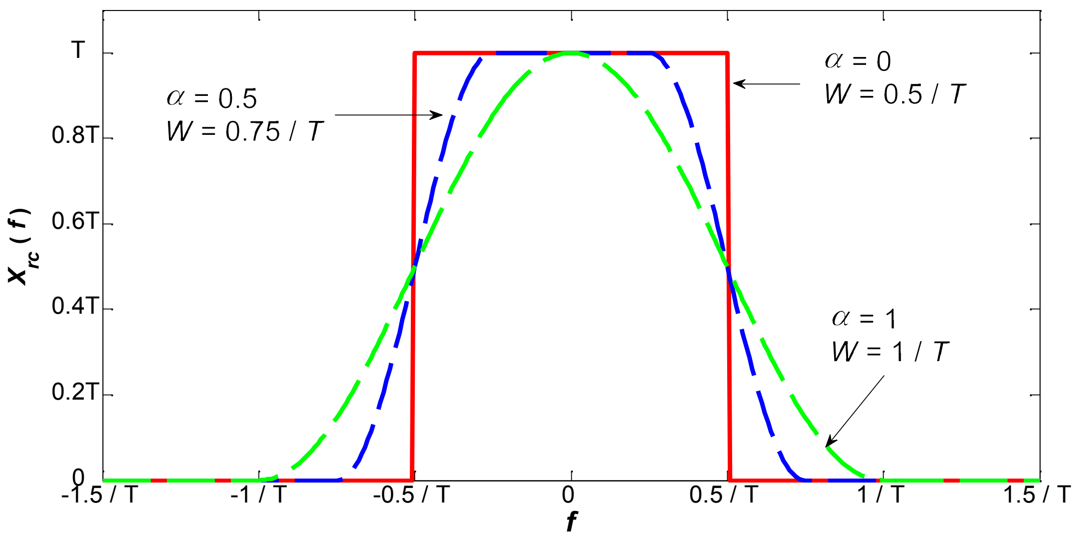
The instance of $t = 0$ also means the shifted sampling instances of $t = mT$.

Now suppose that we have a bandlimited rectangular channel such that

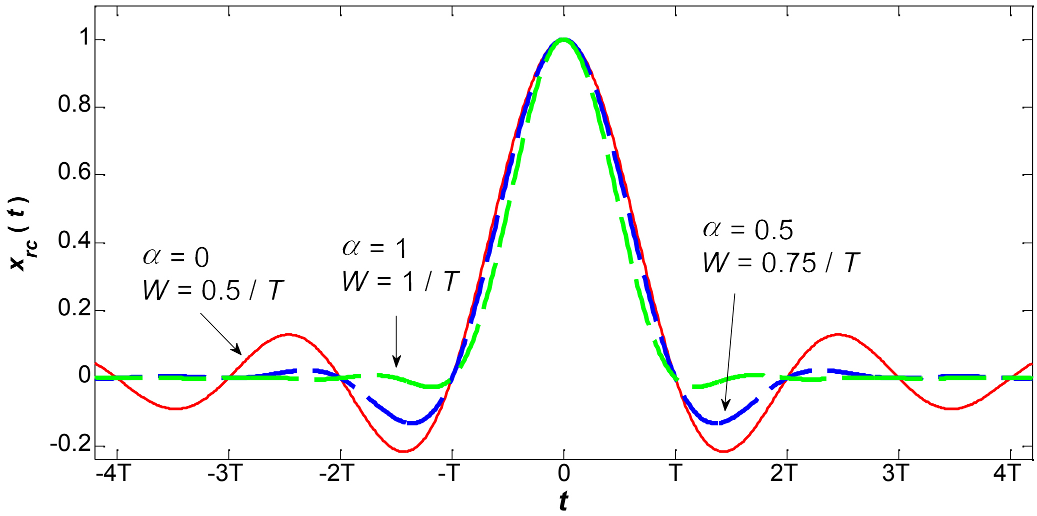
$$C(f) = \begin{cases} 1 & |f| \le W \\ 0 & \text{otherwise} \end{cases} \quad (3.16)$$

Then the raised cosine functionality is shared between the transmitter and receiver filters which means

$$X_{rc}(f) = G_T(f)\,G_R(f) \quad (3.17)$$



a) Plot of $X_{rc}(f)$



a) Plot of $x_{rc}(t)$

Fig. 3.2 The plots of $X_{rc}(f)$ and $x_{rc}(t)$ for $\alpha = 0,\ 0.5,\ 1$.

In particular, if the receiver filter is matched to transmitter filter, i.e. $G_R(f) = G_T^*(f)\exp(-2j\pi f T_s)$, then

$$X_{rc}(f) = \left|G_T(f)\right|^2 \qquad (3.18)$$

(3.18) can alternatively be expressed as

$$G_T(f) = \sqrt{X_{rc}(f)}\exp(-2j\pi f T_s) \qquad (3.19)$$

Under these circumstances, the raised cosine functionality is split evenly between the transmitter and the receiver filters.

4. Signal Design for Partial ISI

From the analysis of previous section, it is seen that in order to have zero ISI, we need $1/T < 2W$. Recognizing that $1/T$ corresponds to symbol rate, we wonder if it is possible to achieve exactly $1/T = 2W$ symbol rate by allowing partial ISI. One such possibility is given below.

$$x(nT) = \begin{cases} 1 & n = 0\ ,\ 1 \\ 0 & \text{otherwise} \end{cases}$$
$$z_n = \begin{cases} T & n = 0\ ,\ -1 \\ 0 & \text{otherwise} \end{cases} \qquad (4.1)$$

The z_n equivalence on the second line of (4.1) is written using (3.8). Now by benefiting from (3.7), we obtain

$$Z(f) = T + T\exp(-j2\pi fT) \tag{4.2}$$

Inserting into $T = 1/2W$ into (4.2) and converting to $X(f)$, we get

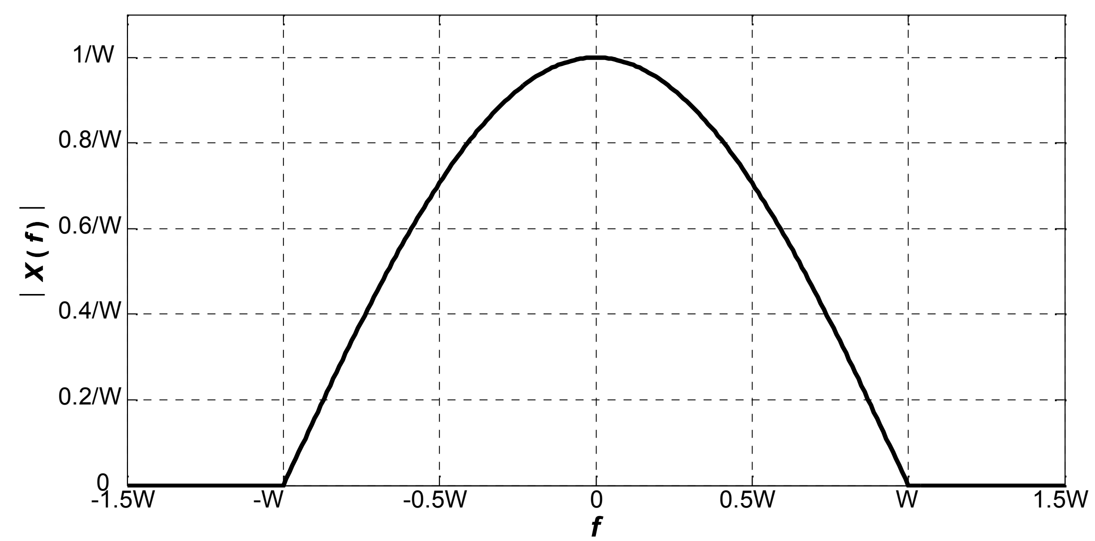
$$\begin{aligned} X(f) &= \begin{cases} \dfrac{1}{2W}\left[1 + T\exp\left(-\dfrac{j\pi f}{W}\right)\right] & |f| \le W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \dfrac{1}{W}\exp\left(-\dfrac{j\pi f}{2W}\right)\cos\left(\dfrac{\pi f}{2W}\right) & |f| \le W \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.3}$$

By inverse Fourier transform, $x(t)$ will be

$$x(t) = \text{sinc}(2Wt) + \text{sinc}(2Wt - 1) \tag{4.4}$$

$X(f)$ and $x(t)$ given in (4.3) and (4.4) are plotted in Fig. 4.1.

In the literature, there exist other partial ISI formulations. Here we terminate this topic.



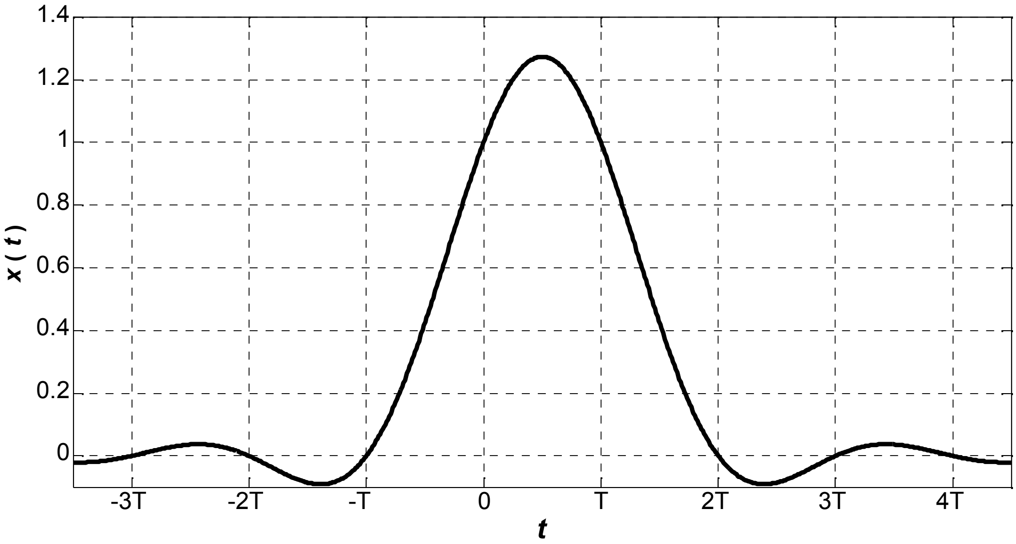


Fig. 4.1 The plots of $X(f)$ and $x(t)$ in partial ISI of (4.3) and (4.4).

5. Signal Design in Presence of Channel Distortion

As proven in the last section, for ISI free transmission over a channel, it must be that

$$X_{rc}(f)\exp(-2j\pi fT_s)=G_T(f)C(f)G_R(f) \tag{5.1}$$

where T_s is a time delay to ensure the physical realizability of the transmitter and receiver filters. Here the critical element is the channel response, since $G_T(f)$ and $G_R(f)$ are in our control. There may be two cases for $C(f)$. The first case would be $C(f)$ does not change with time, so once measured, we gain a knowledge of it. Then our system can safely be designed according (5.1). In the second case, $C(f)$ changes with time, then (5.1) has to be modified.

The characteristics of $C(f)$ can be visualized in terms of amplitude and phase. This means, amplitude distortion of the transmitted signal will occur if $|C(f)|$ is f dependent within the passband of $|f|\leq W$. Similarly if the phase response $\Theta(f)$ is not linear with f, then phase distortion will occur as well. For instance a phase response like the one shown in Fig. 1.2 is a linear one, since it can be expressed as $\Theta(f)=-2\pi fT_s$. In time domain this corresponds to a simple time delay of T_s. On the other hand, the phase response of a RC low pass filter will be nonlinear in f.

Assume that we know the channel response in advance, thus we distribute the raised cosine functionality between the transmitter and receiver filters as

$$G_T(f)=\frac{\sqrt{X_{rc}(f)}}{C(f)}\exp(-2j\pi fT_s)\ \ ,\ \ G_R(f)=\sqrt{X_{rc}(f)} \tag{5.2}$$

At receiver, signal and noise will both pass through the receiver filter with a response of $G_R(f)$. In this configuration, since $G_R(f)$ does not contain any channel dependence, no performance degradation seems to emerge on the receiver side. But on the transmitter side, the transmitter filter $G_T(f)$ is inversely channel response related. Hence if $C(f)<1$ for $|f|\leq W$, then the transmitter filter will have to provide gain compared with the case of $C(f)=1$ for $|f|\leq W$. On the other hand when $C(f)<1$ for $|f|\leq W$, transmitter filter will have to introduce distortion into the transmitted signal. In summary we will experience performance loos, when and if $C(f)<1$ for $|f|\leq W$.

From (1.6), (1.7) and (3.15), for a noise spectral density input of $S_n(f)=N_0/2$, the noise power at the output of the receiver filter will be

$$P_n=\int_{-\infty}^{\infty} S_n(f)\left|G_R(f)\right|^2\ df=\frac{N_0}{2}\int_{-W}^{W} X_{rc}(f)\ df=\frac{N_0}{2} \tag{5.3}$$

Example 5.1 : Determine the transmitter and receiver filter responses for a communication system that transmits a rate of $1/T=4800$ symbols/sec over a channel which has magnitude response of

$$\left|C(f)\right|=\begin{cases}\dfrac{1}{\sqrt{1+(f/W)^2}} & \text{for } |f|\leq W\\ 0 & \text{otherwise}\end{cases} \tag{5.4}$$

where $W=4800$ Hz.

Solution : From the given quantities, we see that $1/T=W$, indicating that a raised cosine filter of (rolloff factor) $\alpha=1$ is to be used. So from (3.13) and (5.2) we have

$$X_{rc}(f)=\frac{T}{2}\left[1+\cos\left(\pi T|f|\right)\right]=T\cos^2\left(\frac{\pi|f|}{9600}\right)$$

$$\left|G_T(f)\right|=\frac{\sqrt{X_{rc}(f)}}{\left|C(f)\right|}=\begin{cases}\sqrt{T\left[1+(f/W)^2\right]}\cos\left(\dfrac{\pi|f|}{9600}\right) & \text{for } |f|\leq W\\ 0 & \text{otherwise}\end{cases}$$

$$G_R(f)=\sqrt{X_{rc}(f)}=\begin{cases}\sqrt{T}\cos\left(\dfrac{\pi|f|}{9600}\right) & \text{for } |f|\leq W\\ 0 & \text{otherwise}\end{cases} \tag{5.5}$$

The plots of $\left|C(f)\right|$, $X_{rc}(f)$, $\left|G_T(f)\right|$ and $G_R(f)$ for Example 5.1 are given below in Fig. 5.1

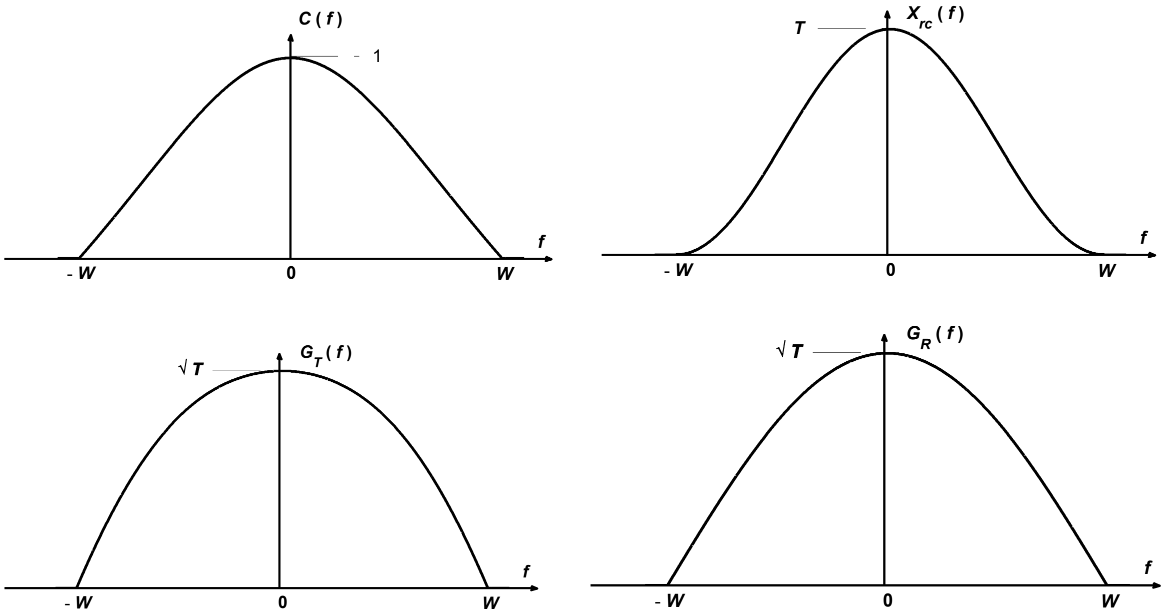


Fig. 5.1 Plots of $|C(f)|$, $X_{rc}(f)$, $|G_T(f)|$ and $G_R(f)$ for Example 5.1.

Exercise 5.1 : Repeat Example 5.1 for the case that the channel has rectangular response, that is

$$C(f) = \begin{cases} 1 & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

Find and make plots of $|C(f)|$, $X_{rc}(f)$, $|G_T(f)|$ and $G_R(f)$ for this case. Compare your findings with those of Example 5.1.

If the channel response is unknown beforehand, thus we measure it on the receiver side (at intervals if required) and incorporate some compensation mechanism of the channel characteristics outside of receiver filter, then we modify (5.2) as follows

$$\begin{aligned} G_T(f) &= \sqrt{X_{rc}(f)} \exp(-2j\pi f T_s) \quad , \quad G_R(f) = \sqrt{X_{rc}(f)} \\ G_E(f) &= \frac{1}{C(f)} = \frac{\exp[-j\Theta(f)]}{|C(f)|} \end{aligned} \tag{5.7}$$

As seen on the second line of (5.7), we have introduced an equalizer with a response of $G_E(f)$ which is set to the inverse of the channel, magnitude and phasewise. Hence $G_E(f)$ equalizes amplitude as well phase distortions. Such an equalizer is known as zero forcing equalizer. The adopted block diagram of a system with an equalizer is shown in Fig. 5.2.

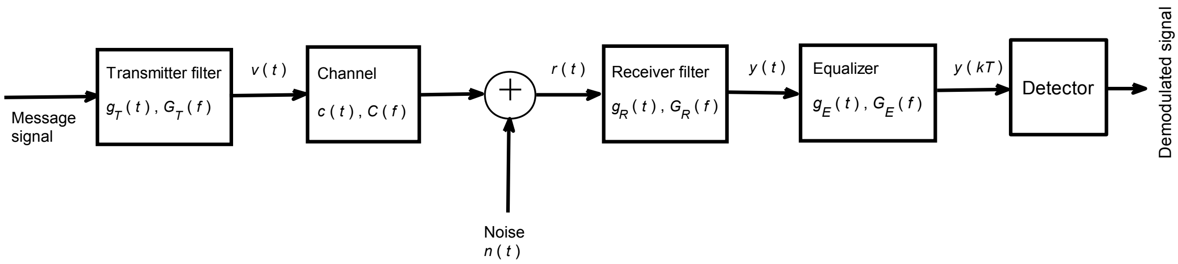


Fig. 5.2 Block diagram of a system with an equalizer.

Now if we carry out an analysis similar to (1.6) and (1.7), for a noise spectral density input of $S_n(f) = N_0/2$, we find the noise power at the output of equalizer as

$$P_n = \int_{-\infty}^{\infty} S_n(f)\left|G_R(f)\right|^2 \left|G_E(f)\right|^2 \, df = \frac{N_0}{2}\int_{-W}^{W} \frac{X_{rc}(f)}{\left|C(f)\right|^2} \, df \qquad (5.8)$$

We know from (3.15) that the integration of the raised cosine function over the whole bandwidth specified by the channel response is unity, but if $C(f) < 1$ for $|f| \le W$, then the noise at the output of zero forcing equalizer will be higher than the case of (5.3) where the channel characteristics are known and hence incorporated into the transmitter filter.

Example 5. 2 : By taking the channel response given in (5.4) of Example 5.1, determine the noise power given in (5.7).

Solution : By inserting into (5.8) for $C(f)$ from (5.4), we have

$$P_n = \frac{N_0}{2}\int_{-W}^{W} T\left[1+\left(f/W\right)^2\right]\cos^2\left(\frac{\pi|f|}{2W}\right) df = N_0\int_{-W}^{W}\left(1+x^2\right)\cos^2\left(\frac{\pi x}{2}\right) dx = 0.565N_0 \qquad (5.9)$$

Note that $0.565N_0$ is slightly larger than $0.5N_0$.

6. Equalizers

Initially we want construct a model of ISI created at receiver of a bandlimited channel. For this, we rewrite the output of the receiver filter from (2.13) as

$$y(t) = \sum_{n=-\infty}^{\infty} a_n x(t-nT) + \eta(t) \qquad (6.1)$$

where $x(t)$ is the cascaded impulse responses of transmitter filter, channel and receiver filter, thus $x(t) = g_T(t) * c(t) * g_R(t)$ and $\eta(t)$ is the noise at the output of the receiver filter, so $\eta(t) = n(t) * g_R(t)$. After sampling at $t = mT$, we have

$$y_m = x_0 a_m + \sum_{\substack{n=-\infty \\ n \neq m}}^{\infty} a_n x_{m-n} + \eta_m \quad , \quad m = 0, \ \pm 1, \ \pm 2 \cdots \tag{6.2}$$

The middle term in (6.2) represents ISI. In a practical system, the summation for this term extends only over finite number of terms. Denoting the lower (negative) limit of this new summation by $-L_1$, the upper limit by L_2, the desired symbol term (first term of (6.2)) and the ISI term can be expressed as

$$x_0 a_m + \sum_{\substack{n=-\infty \\ n \neq m}}^{\infty} a_n x_{m-n} \approx \sum_{k=-L_1}^{L_2} x_k a_{m-k} \quad , \quad L = L_1 + L_2 \tag{6.3}$$

(6.3) can be modelled by discrete time channel filter as shown in Fig. 6.1 (copied from Proakis 2002).

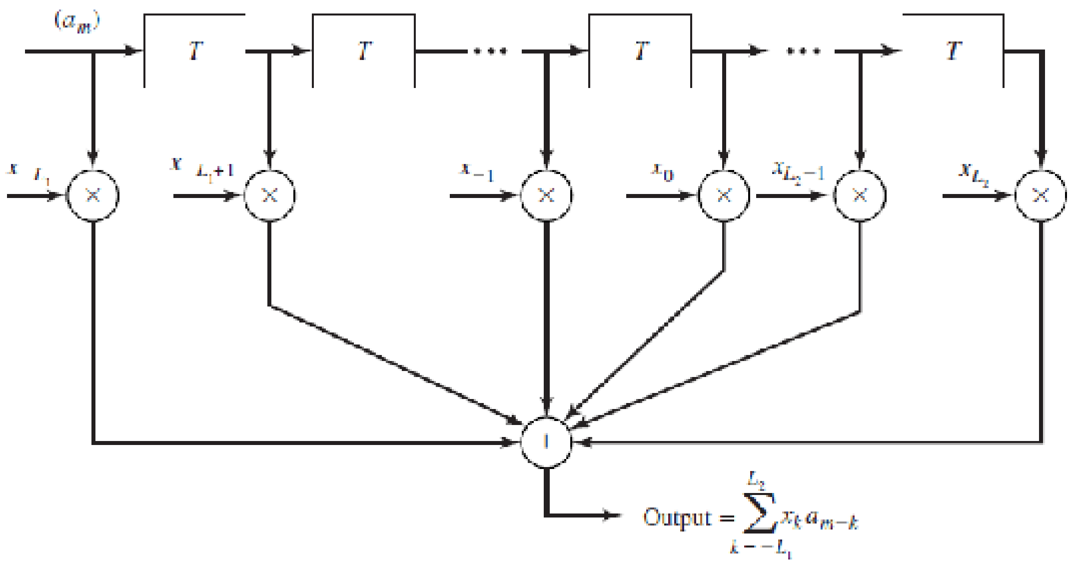


Fig. 6.1 Discrete time channel filter representation.

The boxes in Fig. 6.1 containing the letter T inside are the FIR filters with delays of T each. Now to equalize such an output of receiver filter, we propose an equalizer whose construction is inspired by the representation in Fig. 6.1. As shown in Fig. 6.2 (copied from Proakis 2002), our equalizer will consist of FIR filters whose outputs are multiplied by c_n tap coefficients (only five are shown in Fig. 6.2) and summed, thus the impulse response of the equalizer

$$g_E(t) = \sum_{n=-N}^{N} c_n \delta(t - n\tau) \tag{6.4}$$

The corresponding frequency response of the equalizer is

$$G_E(f)=\sum_{n=-N}^{N} c_n \exp(-j2\pi fn\tau) \tag{6.5}$$

In Fig. 6.2, τ represents the tab spacing, selected to be $\tau < T$, thus said to be at fractional spacing. It is presumed that $2N+1 \geq L$ number of equalizer coefficients are sufficient to span the entire ISI space. As indicated in (2.4) and Fig. 5.1, the signal input to the equalizer is $x(t)$, then with the responses given in (6.4) and (6.5), the output from the equalizer becomes

$$q(t)=\sum_{n=-N}^{N} c_n x(t-n\tau) \tag{6.6}$$

After sampling at $t = mT$ and forcing zero ISI condition, we get

$$q(mT)=\sum_{n=-N}^{N} c_n x(mT-n\tau)\begin{cases}1 & m=0\\ 0 & m=\pm1,\ \pm2\cdots,\ \pm N\end{cases} \tag{6.7}$$

Expressed in a matrix form, (6.7) means

$$\mathbf{q}=\mathbf{Xc} \tag{6.8}$$

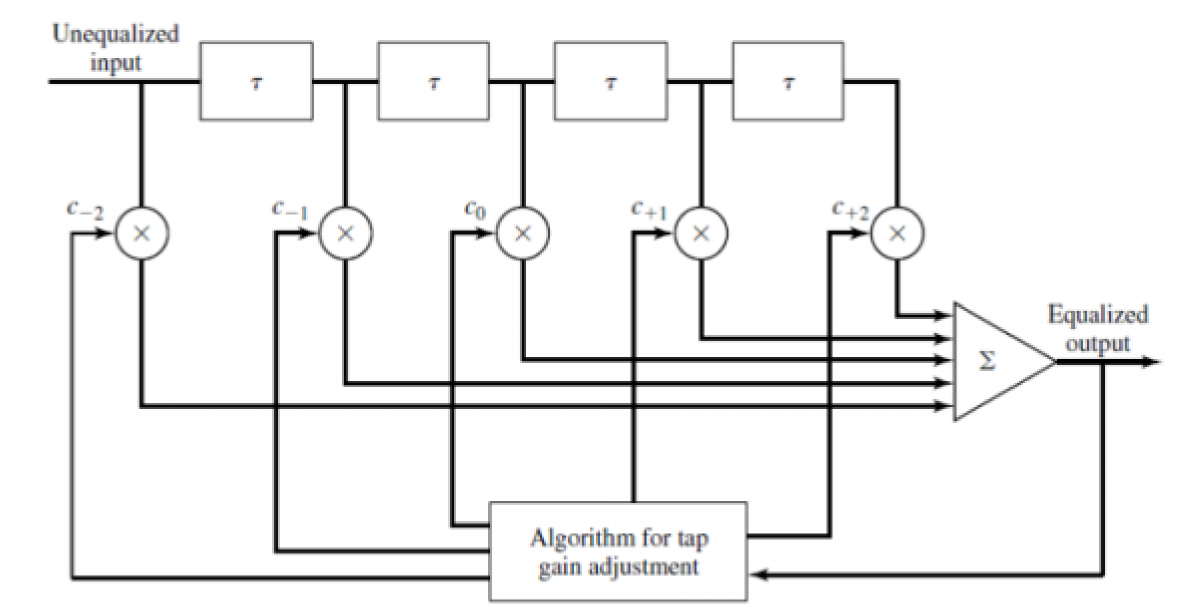


Fig. 6. 2 Diagram of an equalizer with five tabs.

In (6.8), $\mathbf{q}$is a column vector whose upper and lower N rows are zero with the $(N+1)$th row being unity. This way the zero ISI condition (6.7) is satisfied. $\mathbf{X}$ is a matrix of $(2N+1)\times(2N+1)$ obtained by sampling $x(t)$at intervals of $mT-n\tau$. Finally $\mathbf{c}$ is a column vector for the c_n tap coefficients whose values are to be determined according to the zero ISI condition over a symbol span of length $(2N+1)$ from $\mathbf{c}=\mathbf{X}^{-1}\mathbf{q}$.

Example 6.1 A channel distorted pulse of $x(t)$ prior to equalizer is given as follows

$$x(t) = \frac{1}{1+(2t/T)^2} \tag{6.9}$$

The tap spacing is arranged to be at $\tau = T/2$, while the sampling is carried out at $t = mT$. Determine the c_n tap coefficients, i.e. column vector $\mathbf{c}$, if the number of taps is five, i.e. $2N+1=5$.

Solution : To construct $\mathbf{X}$, we use $x(mT - n\tau) = x(mT - nT/2)$ and run m as the row index, n as the column index in the range $-2,\ -1,\ 0,\ 1,\ 2$ and insert such found arguments of $t = mT - nT/2$ into (6.9) and evaluate $x(t)$. This way, $\mathbf{X}$ matrix will become

$$\begin{matrix} n \rightarrow -2 & -1 & 0 & 1 & 2 & m \\ & & & & & \downarrow \end{matrix}$$

$$\mathbf{X} = \begin{pmatrix} \frac{1}{5} & \frac{1}{10} & \frac{1}{17} & \frac{1}{26} & \frac{1}{37} \\ 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{10} & \frac{1}{17} \\ \frac{1}{5} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{17} & \frac{1}{10} & \frac{1}{5} & \frac{1}{2} & 1 \\ \frac{1}{37} & \frac{1}{26} & \frac{1}{17} & \frac{1}{10} & \frac{1}{5} \end{pmatrix} \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{matrix} \tag{6.10}$$

Since a five tab equalizer is requested, then column matrix for $\mathbf{q}$vector is

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{6.11}$$

The tab coefficients then become

$$\mathbf{c} = \mathbf{X}^{-1}\mathbf{q} = \begin{pmatrix} -2.2 \\ 4.9 \\ -3 \\ 4.9 \\ -2.2 \end{pmatrix} \tag{6.12}$$

To check how much equalization is achieved by (6.12), we carry out the following computation which can be found in Q1_FE_26052014.m.

$$\text{The sum of } x(t) \text{ in the range } t=-4T \text{ up to } t=4T : E_{tq} = \sum_{t=-4T}^{4T} x(t,\tau=T/2)$$
$$\text{The sum of } x(t) \text{ in the range (five tab) } t=-T \text{ up to } t=T : E_{eq} = \sum_{t=-T}^{T} x(t,\tau=T/2)$$
$$\text{Equalization Ratio for five tap } R = E_{eq}/E_{eq} = 0.8221$$
$$\text{Equalization Ratio for seven tap } R = E_{eq}/E_{eq} = 0.8906 \tag{6.13}$$

Note that because of the symmetry in (6.9), the tab coefficients in (6.12) have come out to be symmetrical. To compute $\mathbf{X}^{-1}$ from a given $\mathbf{X}$, in Matlab we simply use the operator inv, thus $\mathbf{X}^{-1} = \text{inv}(\mathbf{X})$.

It is easy to see in the use of the equalizer, since $G_E(f) = 1/C(f)$, the equalizer will have to act as a high gain amplifier if $C(f) \ll 1$. Then there will be an accumulation of excess amount of noise. Then we can use an equalization strategy based on minimum mean square error (MMSE) that tries to equalize signal plus noise together rather than signal alone as described above.

Exercise 6. 1 : By changing the number of tap coefficients to $2N+1=3,\ 7$, recalculate $\mathbf{c}$.

Exercise 6. 2 : By changing $x(t)$ to the followings and keeping the number of tap coefficients at five , recalculate $\mathbf{c}$.

$$x(t) = \frac{1}{1+(t/T)^2}$$
$$x(t) = \frac{1}{1+|t|/T} \tag{6.14}$$

Exercise 6. 3 : Now take $\tau = T$ and repeat the calculations in Example 6.1.

Exercise 6. 4 : Bearing in mind the following relations, for the given $x(t)$ in Example 6.1

$$X(f) = G_T(f)C(f)G_R(f) \quad , \quad G_E(f) = \frac{1}{C(f)}$$
$$G_T(f) = \sqrt{X_{rc}(f)}\exp(-2j\pi fT_s) \quad , \quad G_R(f) = \sqrt{X_{rc}(f)}$$
$$G_E(f) = \frac{X_{rc}(f)}{X(f)}\exp(-2j\pi fT_s) \quad , \quad X(f) = F[x(t)] \tag{6.15}$$

find $G_E(f)$ and establish its relevance to (6.5)