Çankaya University – ECE Department – ECE 588 (MT)

Student Name : Date : 21.11.2016
Student Number : Open Source Exam

Questions

1. (70 Points) The constellation diagram of the signal set, $s_1(t)$ to $s_8(t)$ is given in Fig. 1.1.

   a. Identify the type of modulation and dimensionality in this signal set. Write mathematical expression for the time waveforms $s_1(t)$ to $s_8(t)$ and the corresponding basis functions, $\psi_1(t) \cdots \psi_N(t)$ and plot them. Write for the signal vectors $\mathbf{s}_1$ to $\mathbf{s}_8$, and find the distance between signal vector ends.

   b. Draw the demodulator as correlator and matched filter. Assuming that the signal $s_1(t)$ is sent from the transmitted, find the outputs of the correlator and matched filter.

   c. Find the probability of error and decision regions via the evaluations of correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ again assuming $s_1(t)$ was transmitted. Comment on difference of the probability errors of $s_1(t)$ and $s_5(t)$.

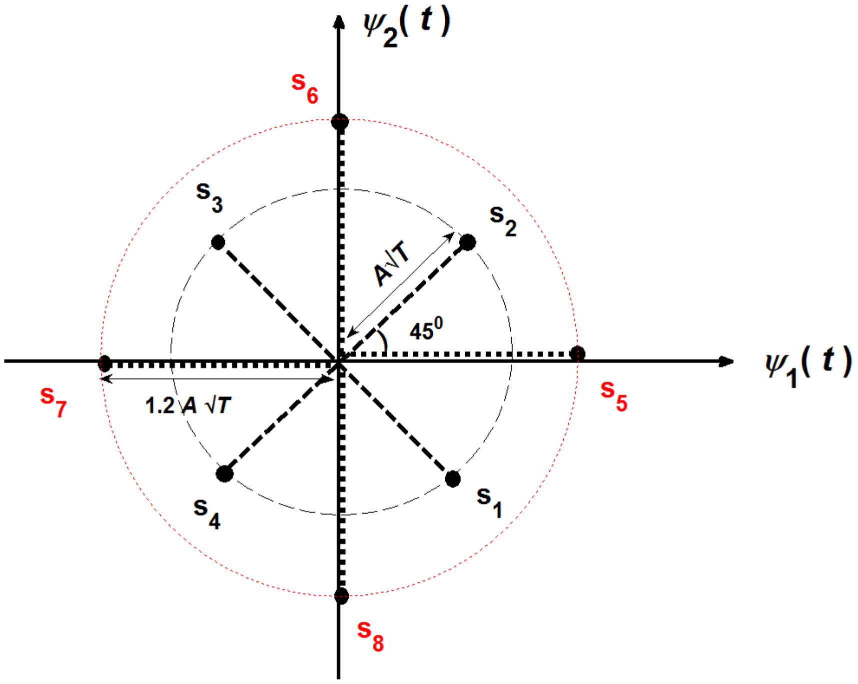

Fig. 1.1 The constellation diagram for Q1.

Solution : a. From the given constellation, it is easy to see that we have 8 QAM where the dimensionality is two.

So we can adapt the following common orthonormalized basis functions,

$$\psi_1(t)=\begin{cases}\sqrt{2/T} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases}\qquad \psi_2(t)=\begin{cases}\sqrt{2/T} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases}\tag{1.1}$$

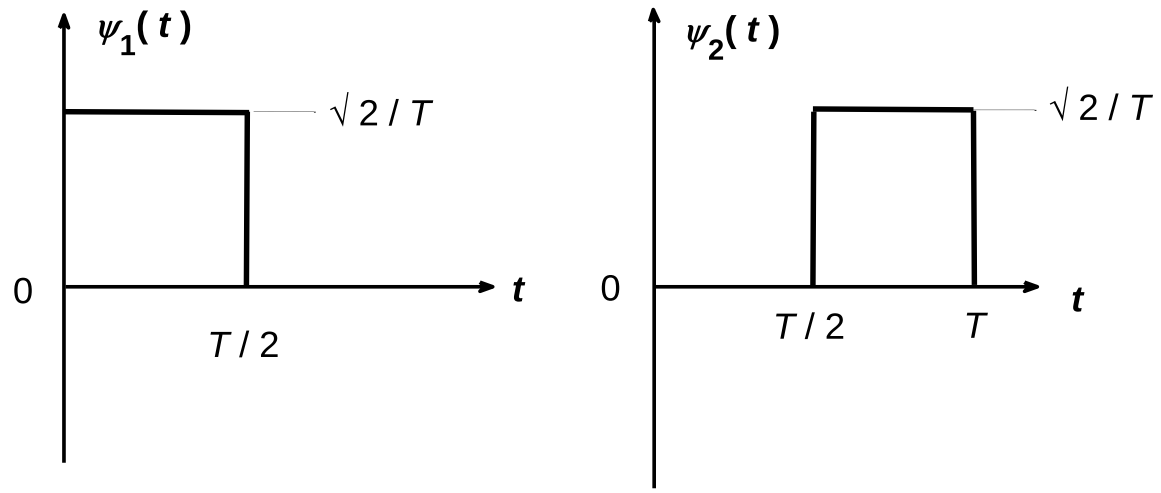

Fig. 1.2 The orthonormalized basis functions for Q1.

By using (1.1), Figs. 1.1 and 1.2, we obtain the followings for the time waveforms of $s_1(t)$ to $s_8(t)$

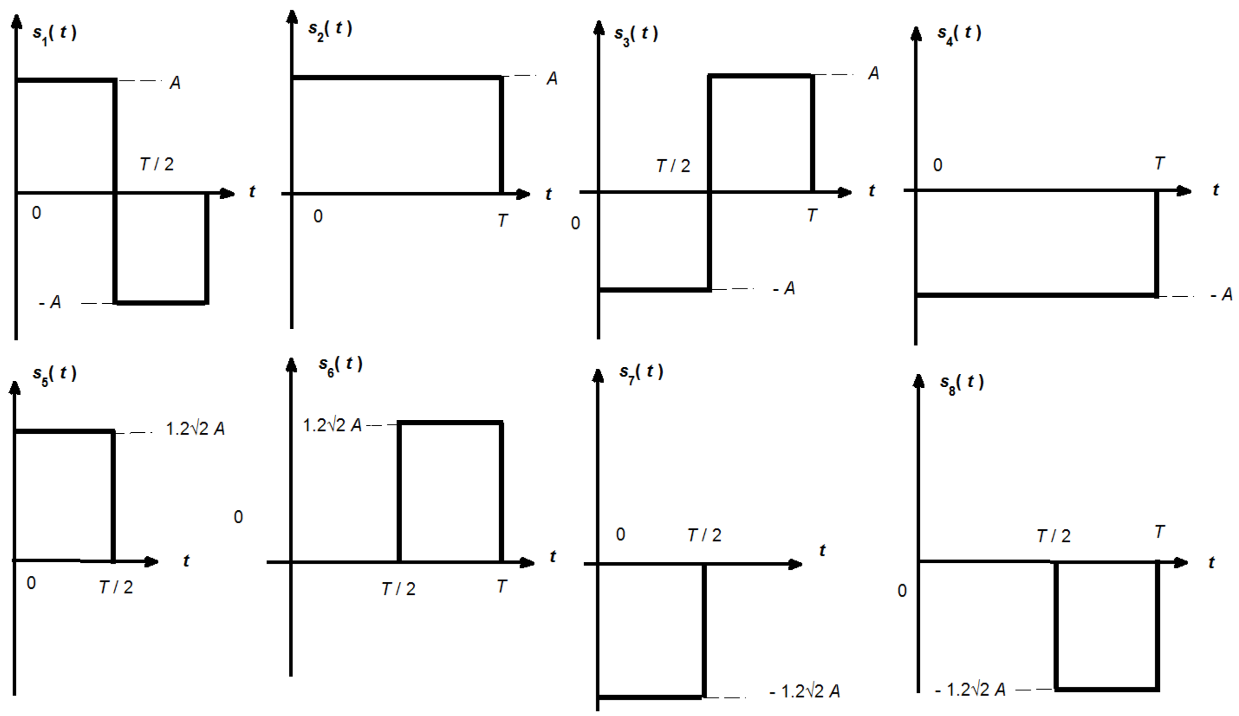

Fig. 1.3 Time waveforms of $s_1(t)\cdots s_8(t)$ for Q1

From Fig. 1.3, it is possible to write the following expressions for $s_1(t)$ to $s_8(t)$

$$s_1(t)=\begin{cases}A & 0\le t\le T/2\\ -A & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_1(t)=A\sqrt{\frac{T}{2}}\psi_1(t)-A\sqrt{\frac{T}{2}}\psi_2(t),\ \mathbf{s_1}=[s_{11},\ s_{12}]=\left[A\sqrt{\frac{T}{2}},\ -A\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_1}=\|\mathbf{s_1}\|^2=A^2T$$

$$s_2(t)=\begin{cases}A & 0\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_2(t)=A\sqrt{\frac{T}{2}}\psi_1(t)-A\sqrt{\frac{T}{2}}\psi_2(t),\ \mathbf{s_2}=[s_{21},\ s_{22}]=\left[A\sqrt{\frac{T}{2}},\ A\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_2}=\|\mathbf{s_2}\|^2=A^2T$$

$$s_3(t)=\begin{cases}-A & 0\le t\le T/2\\ A & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_3(t)=-A\sqrt{\frac{T}{2}}\psi_1(t)+A\sqrt{\frac{T}{2}}\psi_2(t),\ \mathbf{s_3}=[s_{31},\ s_{32}]=\left[-A\sqrt{\frac{T}{2}},\ A\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_3}=\|\mathbf{s_3}\|^2=A^2T$$

$$s_4(t)=\begin{cases}-A & 0\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_4(t)=-A\sqrt{\frac{T}{2}}\psi_1(t)-A\sqrt{\frac{T}{2}}\psi_2(t),\ \mathbf{s_4}=[s_{41},\ s_{42}]=\left[-A\sqrt{\frac{T}{2}},\ -A\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_4}=\|\mathbf{s_4}\|^2=A^2T$$

$$s_5(t)=\begin{cases}1.2\sqrt{2}A & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases},\ s_5(t)=1.2A\sqrt{T}\psi_1(t),\ \mathbf{s_5}=[s_{51},\ s_{52}]=\left[1.2A\sqrt{T},\ 0\right],\ \varepsilon_{s_5}=1.44A^2T$$

$$s_6(t)=\begin{cases}1.2\sqrt{2}A & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_6(t)=1.2A\sqrt{T}\psi_2(t),\ \mathbf{s_6}=[s_{61},\ s_{62}]=\left[0,\ 1.2A\sqrt{T}\right],\ \varepsilon_{s_6}=1.44A^2T$$

$$s_7(t)=\begin{cases}-1.2\sqrt{2}A & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases},\ s_7(t)=-1.2A\sqrt{T}\psi_1(t),\ \mathbf{s_7}=[s_{71},\ s_{72}]=\left[-1.2A\sqrt{T},\ 0\right],\ \varepsilon_{s_7}=1.44A^2T$$

$$s_8(t)=\begin{cases}-1.2\sqrt{2}A & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_8(t)=-1.2A\sqrt{T}\psi_2(t),\ \mathbf{s_8}=[s_{81},\ s_{82}]=\left[0,\ -1.2A\sqrt{T}\right],\ \varepsilon_{s_8}=1.44A^2T$$

$$d_{12}=d_{14}=d_{34}=d_{23}=\sqrt{2}A\sqrt{T}=\sqrt{2\varepsilon_{s_1}},\ d_{13}=d_{24}=2A\sqrt{T}=2\sqrt{\varepsilon_{s_1}}$$

$$d_{56}=d_{67}=d_{78}=d_{58}=1.2A\sqrt{2T}=\sqrt{2\varepsilon_{s_5}},\ d_{57}=d_{68}=2.4A\sqrt{T}=2\sqrt{\varepsilon_{s_5}}$$

Min distance between inner and outer circle, $d_{25}=1.3164A\sqrt{T}$ (min distance of the constellation)

$$|\mathbf{s}_1|=|\mathbf{s}_2|=|\mathbf{s}_3|=|\mathbf{s}_4|=A\sqrt{T}=\sqrt{\varepsilon_{s_1}}\quad,\quad |\mathbf{s}_5|=|\mathbf{s}_6|=|\mathbf{s}_7|=|\mathbf{s}_8|=1.2A\sqrt{T}=1.2\sqrt{\varepsilon_{s_1}}$$

$$\mathbf{s_1}=\left[\sqrt{\frac{\varepsilon_{s_1}}{2}},\ -\sqrt{\frac{\varepsilon_{s_1}}{2}}\right],\ \mathbf{s_2}=\left[\sqrt{\frac{\varepsilon_{s_1}}{2}},\ \sqrt{\frac{\varepsilon_{s_1}}{2}}\right],\ \mathbf{s_3}=\left[-\sqrt{\frac{\varepsilon_{s_1}}{2}},\ \sqrt{\frac{\varepsilon_{s_1}}{2}}\right],\ \mathbf{s_4}=\left[-\sqrt{\frac{\varepsilon_{s_1}}{2}},\ -\sqrt{\frac{\varepsilon_{s_1}}{2}}\right]$$

$$\mathbf{s_5}=\left[1.2\sqrt{\varepsilon_{s_1}},\ 0\right],\ \mathbf{s_6}=\left[0,\ 1.2\sqrt{\varepsilon_{s_1}}\right],\ \mathbf{s_7}=\left[-1.2\sqrt{\varepsilon_{s_1}},\ 0\right],\ \mathbf{s_8}=\left[0,\ -1.2\sqrt{\varepsilon_{s_1}}\right] \qquad (1.2)$$

b. Since QAM is two dimensional, for block diagrams of correlator and MF, we benefit from Fig. 6.7 of Notes on Dimensionality of Signals_Sept 2012_HTE, which is not reproduced here to save space. Below we just give the outputs.

If $s_1(t)$ is sent from the transmitter in then the output from the upper and lower braches of the correlator will be after sampling will be

$$y_1 = \int_0^T r(t)\ \psi_1(t)dt = \int_0^T s_1(t)\ \psi_1(t)dt + \int_0^T n(t)\ \psi_1(t)dt = s_{11} + n_1 = A\sqrt{\frac{T}{2}} + n_1$$
$$y_2 = \int_0^T r(t)\ \psi_2(t)dt = \int_0^T s_1(t)\ \psi_2(t)dt + \int_0^T n(t)\ \psi_2(t)dt = s_{12} + n_2 = -A\sqrt{\frac{T}{2}} + n_2 \qquad (1.3)$$

We know that output from MF will be identical to (1.3) at the time of sampling at $t = T$ which means that we can construct the received vector $\mathbf{r}$ that we supply to the detector and which will be used in the decision making process, as follows

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} A\sqrt{\frac{T}{2}} + n_1 \\ -A\sqrt{\frac{T}{2}} + n_2 \end{bmatrix} \qquad (1.4)$$

c. Using (1.4), we evaluate correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ for $m = 1 \cdots 8$ as follows

$$C(\mathbf{r}, \mathbf{s_m}) = 2\,\mathbf{s_m}\cdot\mathbf{r} - \|\mathbf{s_m}\|^2 \;\;,\quad m = 1\cdots 8$$

$$m=1\;,\; C(\,\mathbf{r}, \mathbf{s_1}) = 2\,\mathbf{s_1}\cdot\mathbf{r} - \|\mathbf{s_1}\|^2 = 2\left[A\sqrt{\frac{T}{2}},\; -A\sqrt{\frac{T}{2}}\right]\begin{bmatrix} A\sqrt{\frac{T}{2}}+n_1 \\ -A\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} - A^2T = A\sqrt{T}\left(\sqrt{2}n_1 - \sqrt{2}n_1 + A\sqrt{T}\right)$$

$$m=2\;,\; C(\,\mathbf{r}, \mathbf{s_2}) = 2\,\mathbf{s_2}\cdot\mathbf{r} - \|\mathbf{s_2}\|^2 = 2\left[A\sqrt{\frac{T}{2}},\; A\sqrt{\frac{T}{2}}\right]\begin{bmatrix} A\sqrt{\frac{T}{2}}+n_1 \\ -A\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} - A^2T = A\sqrt{T}\left(\sqrt{2}n_1 + \sqrt{2}n_1 - A\sqrt{T}\right)$$

$$m=3\;,\; C(\,\mathbf{r}, \mathbf{s_3}) = 2\,\mathbf{s_3}\cdot\mathbf{r} - \|\mathbf{s_3}\|^2 = 2\left[-A\sqrt{\frac{T}{2}},\; A\sqrt{\frac{T}{2}}\right]\begin{bmatrix} A\sqrt{\frac{T}{2}}+n_1 \\ -A\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} - A^2T = -A\sqrt{T}\left(\sqrt{2}n_1 - \sqrt{2}n_1 + 3A\sqrt{T}\right)$$

$$m=4\;,\; C(\,\mathbf{r}, \mathbf{s_4}) = 2\,\mathbf{s_4}\cdot\mathbf{r} - \|\mathbf{s_4}\|^2 = 2\left[-A\sqrt{\frac{T}{2}},\; -A\sqrt{\frac{T}{2}}\right]\begin{bmatrix} A\sqrt{\frac{T}{2}}+n_1 \\ -A\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} - A^2T = -A\sqrt{T}\left(\sqrt{2}n_1 + \sqrt{2}n_1 + A\sqrt{T}\right)$$

$$m=5\;,\; C(\,\mathbf{r}, \mathbf{s_5}) = 2\,\mathbf{s_5}\cdot\mathbf{r} - \|\mathbf{s_5}\|^2 = 2\left[1.2A\sqrt{T},\; 0\right]\begin{bmatrix} A\sqrt{\frac{T}{2}}+n_1 \\ -A\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} - 1.44A^2T = \frac{6}{25}A\sqrt{T}\left[10n_1 + \left(5\sqrt{2}-6\right)A\sqrt{T}\right]$$

$$m=6\;,\; C(\,\mathbf{r}, \mathbf{s_6}) = 2\,\mathbf{s_6}\cdot\mathbf{r} - \|\mathbf{s_6}\|^2 = 2\left[0,\; 1.2A\sqrt{T}\right]\begin{bmatrix} A\sqrt{\frac{T}{2}}+n_1 \\ -A\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} - 1.44A^2T = -\frac{6}{25}A\sqrt{T}\left[-10n_2 + \left(5\sqrt{2}+6\right)A\sqrt{T}\right]$$

$$m=7\;,\; C(\,\mathbf{r}, \mathbf{s_7}) = 2\,\mathbf{s_7}\cdot\mathbf{r} - \|\mathbf{s_7}\|^2 = 2\left[-1.2A\sqrt{T},\; 0\right]\begin{bmatrix} A\sqrt{\frac{T}{2}}+n_1 \\ -A\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} - 1.44A^2T = -\frac{6}{25}A\sqrt{T}\left[10n_1 + \left(5\sqrt{2}+6\right)A\sqrt{T}\right]$$

$$m=8\;,\; C(\,\mathbf{r}, \mathbf{s_8}) = 2\,\mathbf{s_8}\cdot\mathbf{r} - \|\mathbf{s_8}\|^2 = 2\left[-1.2A\sqrt{T},\; 0\right]\begin{bmatrix} A\sqrt{\frac{T}{2}}+n_1 \\ -A\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} - 1.44A^2T = -\frac{6}{25}A\sqrt{T}\left[10n_2 - \left(5\sqrt{2}-6\right)A\sqrt{T}\right] \quad (1.6)$$

The correct decision region for $\mathbf{s}_1$ for the case of $s_1(t)$ being transmitted is determined by the following inequalities and corresponding conditions.

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_2) : 0 > \overbrace{-A\sqrt{\frac{T}{2}}+n_2}^{r_2} \rightarrow r_2 < 0$$

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_3) : \overbrace{A\sqrt{\frac{T}{2}}+n_1}^{r_1} > \overbrace{-A\sqrt{\frac{T}{2}}+n_2}^{r_2} \rightarrow r_1 > r_2$$

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_4) : \overbrace{A\sqrt{\frac{T}{2}}+n_1}^{r_1} > 0 \rightarrow r_1 > 0$$

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_5) : \left(30-\frac{25}{\sqrt{2}}\right)n_1 + \frac{25}{\sqrt{2}}n_2 - \left(\frac{61}{\sqrt{2}}-30\right)A\sqrt{\frac{T}{2}} > 0 \rightarrow r_2 < \left(1-\frac{30\sqrt{2}}{25}\right)r_1 + \frac{11}{25\sqrt{2}}\overbrace{\sqrt{\varepsilon_{s_1}}}^{A\sqrt{T}}$$

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_6) : \left(30+\frac{25}{\sqrt{2}}\right)n_2 + \frac{25}{\sqrt{2}}n_1 + \left(\frac{61}{\sqrt{2}}+30\right)A\sqrt{\frac{T}{2}} > 0 \rightarrow r_2 < \frac{25r_1}{30\sqrt{2}+25} + \frac{11}{30\sqrt{2}+25}\overbrace{\sqrt{\varepsilon_{s_1}}}^{A\sqrt{T}}$$

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_7) : \left(30+\frac{25}{\sqrt{2}}\right)n_1 - \frac{25}{\sqrt{2}}n_2 + \left(\frac{61}{\sqrt{2}}+30\right)A\sqrt{\frac{T}{2}} > 0 \rightarrow r_2 < \frac{30\sqrt{2}+25}{25}r_1 + \frac{11}{25\sqrt{2}}\overbrace{\sqrt{\varepsilon_{s_1}}}^{A\sqrt{T}}$$

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_8) : \left(30-\frac{25}{\sqrt{2}}\right)n_2 + \frac{25}{\sqrt{2}}n_1 + \left(\frac{61}{\sqrt{2}}-30\right)A\sqrt{\frac{T}{2}} > 0 \rightarrow r_2 > \frac{-25r_1}{30\sqrt{2}-25} - \frac{11}{60-25\sqrt{2}}\overbrace{\sqrt{\varepsilon_{s_1}}}^{A\sqrt{T}} \quad (1.7)$$

Note that the above evaluations are carried out in the m file ECE588_MT2016_Q1Calculations.m both in terms of $A,\ T,\ n_1,\ n_2$ and also $r_1,\ r_2,\ \varepsilon_{s_1}$.

The related graphs and the decision regions of (1.7) can be found in Figs. 1.4, 1.5 and 1.6. Trying to find the common region of all inequalities listed in (1.7) and graphically illustrated in Figs. 1.4 to 1.6, we get Fig. 1.7. As seen from this figure, that common region is bordered by

$$C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_2),\ C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_4),\ C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_5),\ C(\mathbf{r},\mathbf{s}_1) > C(\mathbf{r},\mathbf{s}_8) \quad (1.8)$$

This means that the other inequalities are already covered by the ones given in (1.8).

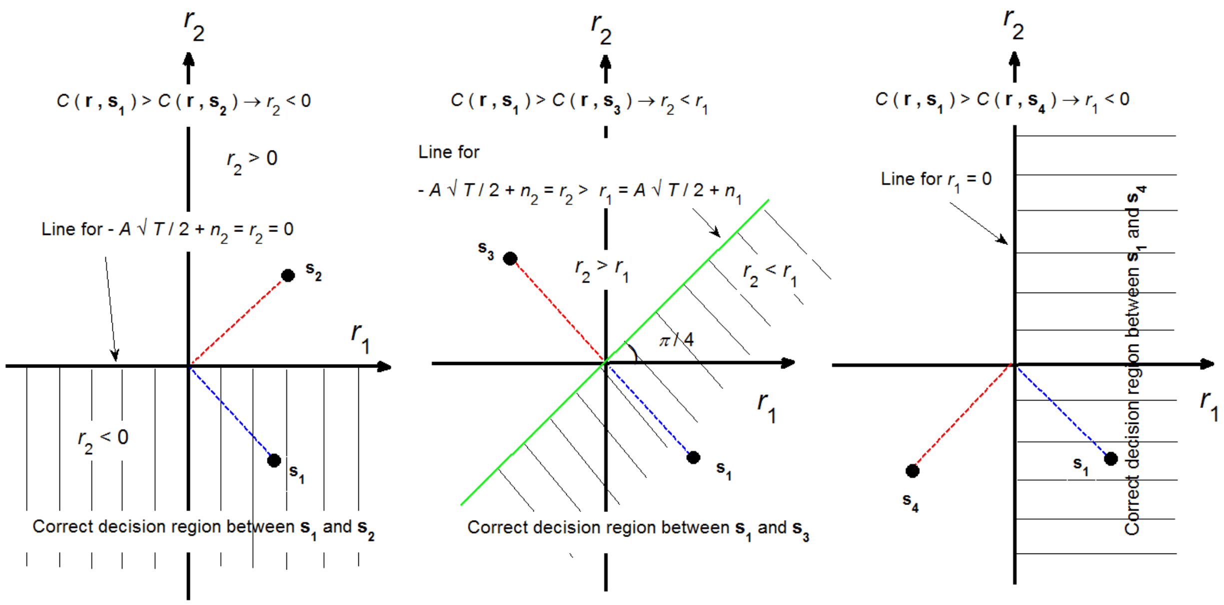

Fig. 1. 4 Decision regions for $\mathbf{s}_1$ against $\mathbf{s}_2$, $\mathbf{s}_3$ and $\mathbf{s}_4$.

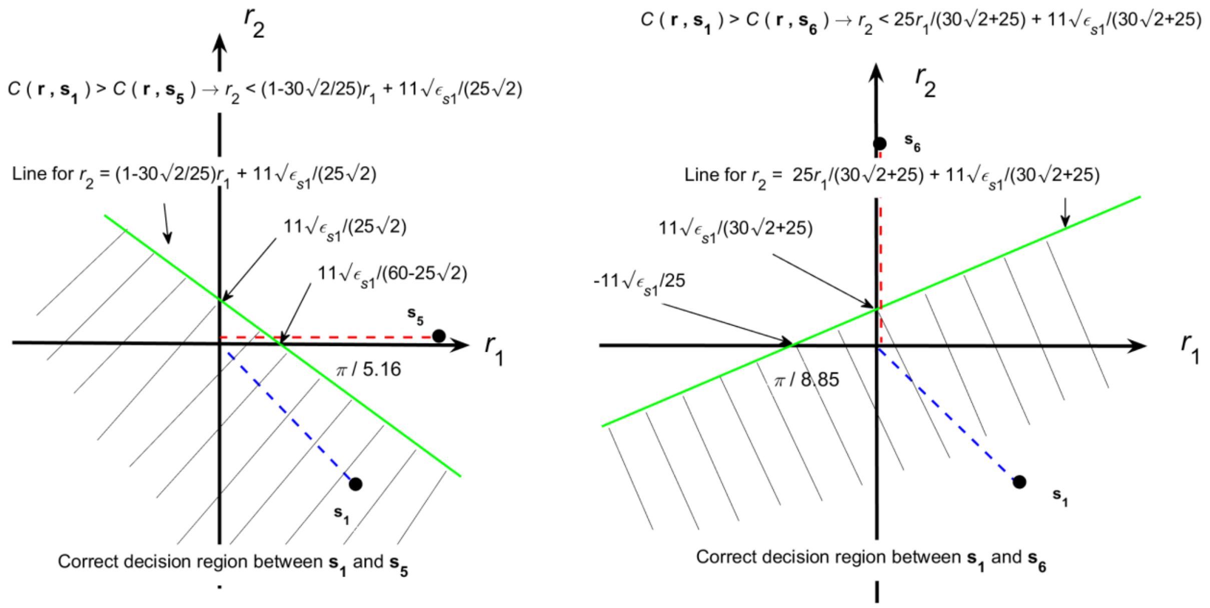

Fig. 1. 5 Decision regions for $\mathbf{s}_1$ against $\mathbf{s}_5$ and $\mathbf{s}_6$.

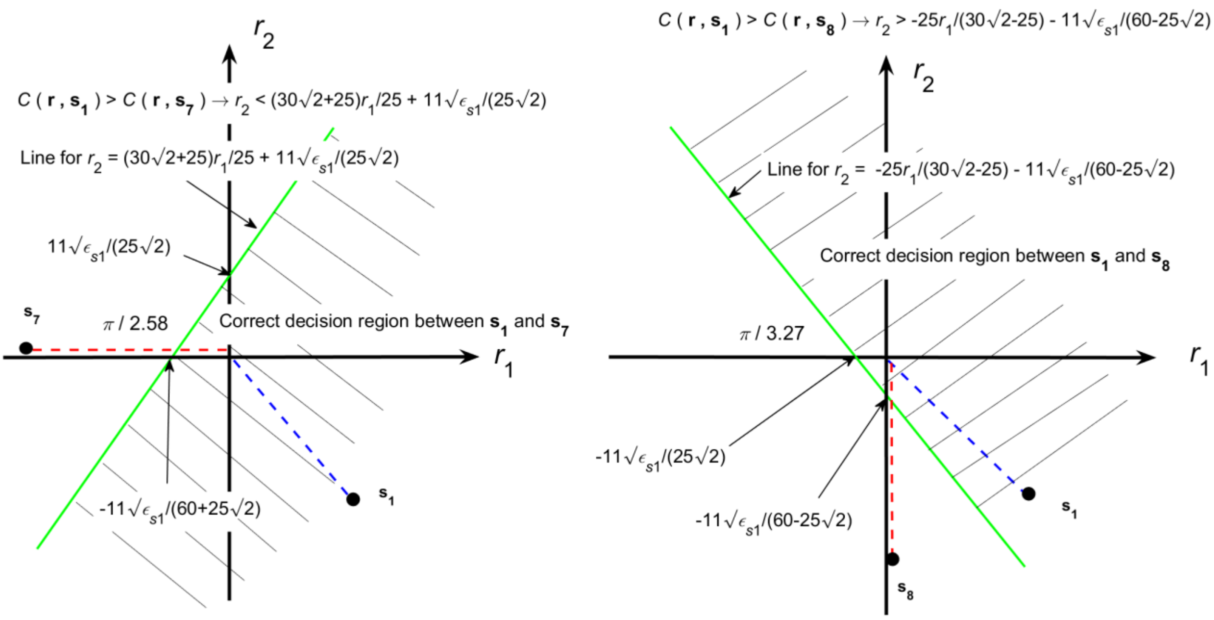

Fig. 1. 6 Decision regions for $\mathbf{s}_1$ against $\mathbf{s}_7$ and $\mathbf{s}_8$.

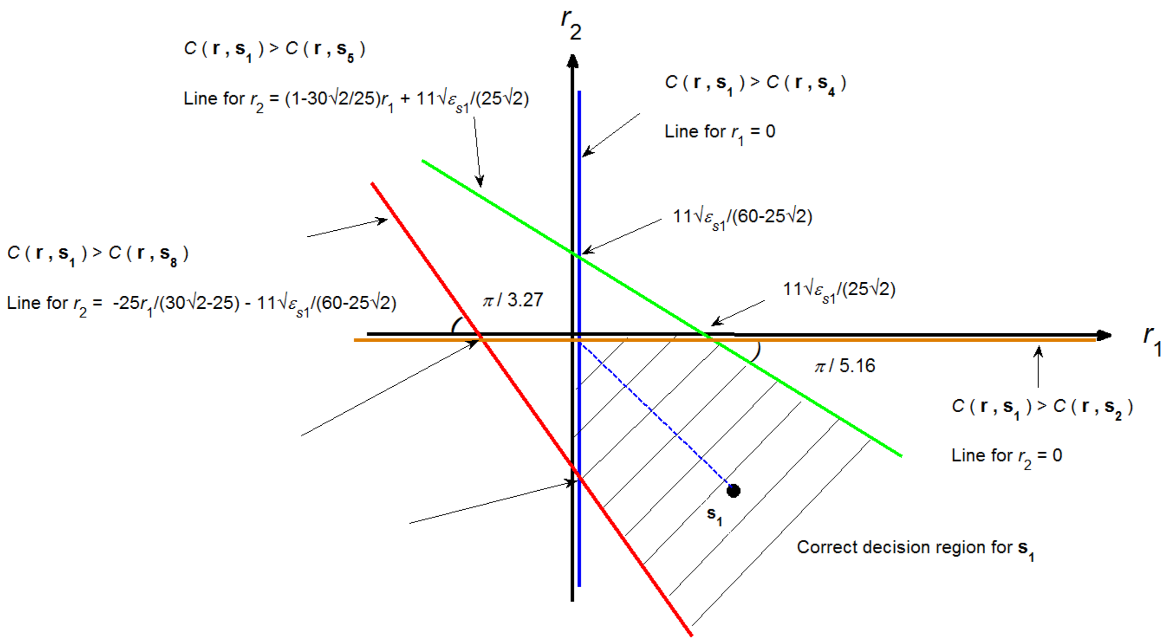

Fig. 1. 7 Combined decision region for $\mathbf{s}_1$.

To find the probability of error for $\mathbf{s}_1$, we need to integrate in the shaded area of Fig. 1.7, which is left as an exercise.

2. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer

   a) For a baseband channel limited to $-W \le f \le W$, the maximum achievable ISI free symbol rate is $R_s = \frac{1}{T} = 2W$, where T is the symbol duration : From Fig. 3.2 of lecture notes entitled, Notes on ISI_Sept 2012_HTE and from (the solution of) Q3 of Sample ISI Problems from Proakis 2002_2012, we have $W = \frac{1+\alpha}{2T} \to R_s = \frac{2W}{1+\alpha} \to 2W\big|_{\alpha=0}$. So the answer is true, but this case is not preferred, since it requires the use of sinc pulses whose tails do not converge rapidly as explained in Notes on ISI_Sept 2012_HTE.

   b) As the rolloff factor, α in the raised cosine filter of $X_{rc}(f)$ is increased, the maximum ISI free symbol rate achievable through an bandlimited channel increases : On the contrary, as illustrated in Fig. 3.2a of lecture notes entitled, "Notes on ISI_Sept 2012_HTE" and as explained in a), at a rooloff factor of $\alpha = 0$, our symbol rate, is $R_s = \frac{1}{T} = 2W$, but at $\alpha = 1$, it is reduced to $R_s = \frac{1}{T} = W$. So the answer is false.

   c) Channel equalization is applied to unlimited channels : False, we have no ISI problem in unlimited, therefore no need to apply channel equalization in unlimited channels. So the answer is false.

   d) As the number of taps in a channel equalizer increases, we get better channel equalization : True as illustrated in (6.13) of lecture notes entitled, "Notes on ISI_Sept 2012_HTE".

   e) To achieve zero ISI over a bandlimited channel, we use sinc pulses in the transmitter : Partially true, we can use sinc pulses to eliminate ISI, but raised cosine function is a better choice as explained in, lecture notes entitled, "Notes on ISI_Sept 2012_HTE".

   f) To achieve zero ISI over a bandlimited channel, we use raised cosine pulses in the receiver : False, since the raised cosine functionality is shared between the transmitter and receiver filters and the pulses (symbols) sent from the transmitter can actually be in the shape of raised cosine function.