# Çankaya University – ECE Department – ECE 588 (MT)

Student Name :
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Open Source Exam

Questions

1. (70 Points) The time waveforms of the signal set, $s_1(t) \cdots s_4(t)$ are given in Fig. 1.1.

   a. Identify the type of modulation and dimensionality in this signal set. Write mathematical expression for $s_1(t) \cdots s_4(t)$ and the corresponding basis functions, $\psi_1(t) \cdots \psi_N(t)$ and plot $\psi_1(t) \cdots \psi_N(t)$. Write for the signal vectors $\mathbf{s}_1 \cdots \mathbf{s}_4$, and plot the corresponding constellation diagram. Find the distance between signal vector ends.
   
   b. Draw the demodulator as correlator and matched filter. Assuming that the signal $s_1(t)$ from constellation is transmitted, find the outputs of the correlator and matched filter.
   
   c. Find the probability of error and decision regions via the evaluations of correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$ again assuming $s_1(t)$ was transmitted. Comment on whether you would find any probability of error performance difference if another signal from the set $s_2(t) \cdots s_4(t)$ was transmitted.

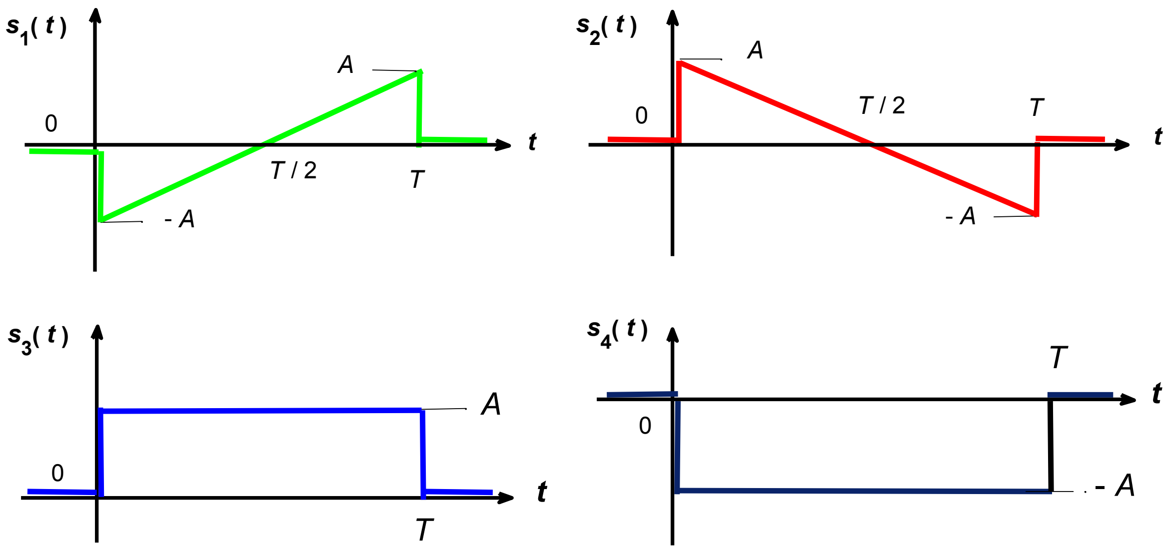


Fig. 1.1 The time waveforms, $s_1(t) \cdots s_4(t)$ for Q1.

**Solution :** a. From Fig. 1.1, we see that the energies of signals are unequal, hence the given time waveforms $s_1(t) \cdots s_4(t)$ will constitute 4 QAM, if they can be represented by two orthonormal basis functions. For these tests we write

$$\varepsilon_{s_1} = \int_{-\infty}^{\infty} s_1^2(t)\, dt = \int_0^T s_1^2(t)\, dt = A^2 \int_0^T \left(\frac{2t}{T} - 1\right)^2 dt = \frac{A^2 T}{3} = \varepsilon_{s_2}$$

$$\varepsilon_{s_3} = \int_{-\infty}^{\infty} s_3^2(t)\, dt = \int_0^T s_3^2(t)\, dt = A^2 \int_0^T dt = A^2 T = \varepsilon_{s_4} \quad , \quad \varepsilon_{s_1} = \varepsilon_{s_2} \neq \varepsilon_{s_3} = \varepsilon_{s_4} \qquad (1.1)$$

With eye inspection, we can devise the following orthonormalized basis functions for the signal set, $s_1(t) \cdots s_4(t)$

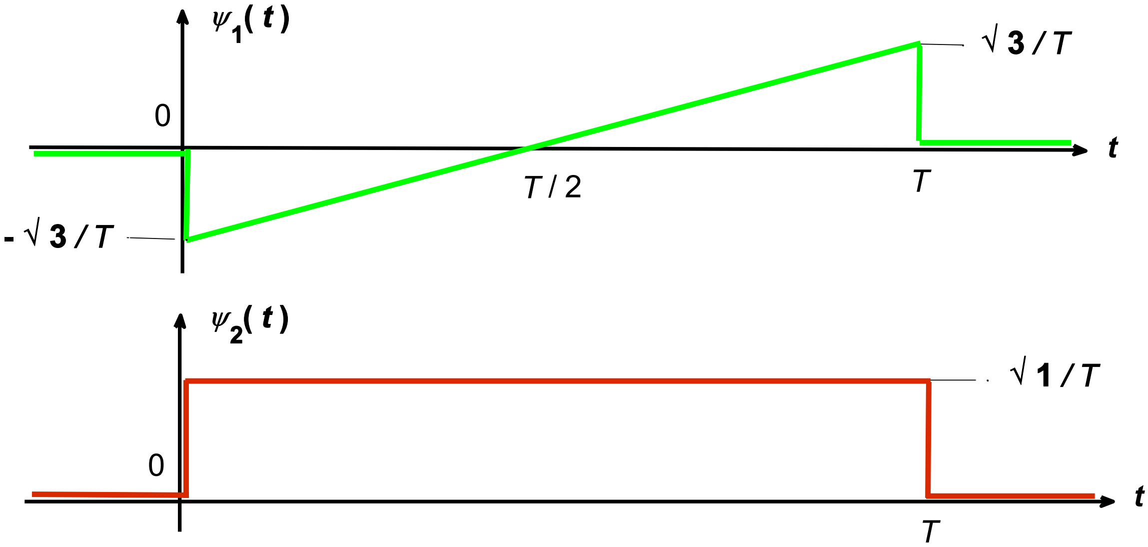


Fig. 1.2 Orthonormalized basis functions, $\psi_1(t)$, $\psi_2(t)$ for the signal set, $s_1(t) \cdots s_4(t)$.

The time waveform expressions for $\psi_1(t)$, $\psi_2(t)$ are

$$\psi_1(t) = \begin{cases} \sqrt{3/T}\left(\frac{2t}{T} - 1\right) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \quad , \quad \psi_1(t) = \begin{cases} \sqrt{1/T} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad (1.2)$$

It is easy to verify that

$$\int_{-\infty}^{\infty} \psi_1(t)\, \psi_2(t) dt = \int_0^T \psi_1(t)\, \psi_2(t) dt = 0$$

$$\int_{-\infty}^{\infty} \psi_1^2(t)\, dt = \int_0^T \psi_1^2(t)\, dt = \int_{-\infty}^{\infty} \psi_2^2(t)\, dt = \int_0^T \psi_2^2(t)\, dt = 1 \qquad (1.3)$$

The time waveform expression for $s_1(t) \cdots s_4(t)$ and in terms of $\psi_1(t)$, $\psi_2(t)$, the signal vector representations for $\mathbf{s}_1 \cdots \mathbf{s}_4$ are given in (1.4). The corresponding constellation diagram is provided in Fig. 1.3.

$$s_1(t)=\begin{cases}A\left(\frac{2t}{T}-1\right) & 0\le t\le T\\ 0 & \text{otherwise}\end{cases} \quad , \quad s_2(t)=\begin{cases}A\left(1-\frac{2t}{T}\right) & 0\le t\le T\\ 0 & \text{otherwise}\end{cases}$$

$$s_3(t)=\begin{cases}A & 0\le t\le T\\ 0 & \text{otherwise}\end{cases} \quad , \quad s_4(t)=\begin{cases}-A & 0\le t\le T\\ 0 & \text{otherwise}\end{cases}$$

$$s_1(t)=A\sqrt{T/3}\psi_1(t) \ , \ s_2(t)=-A\sqrt{T/3}\psi_1(t) \ , \ s_3(t)=A\sqrt{T}\psi_2(t) \ , \ s_4(t)=-A\sqrt{T}\psi_2(t)$$

$$\mathbf{s}_1=[s_{11},\ s_{12}]=\left[A\sqrt{T/3},\ 0\right] \ , \ \mathbf{s}_2=[s_{21},\ s_{22}]=\left[-A\sqrt{T/3},\ 0\right]$$

$$\mathbf{s}_3=[s_{31},\ s_{32}]=\left[0,\ A\sqrt{T}\right] \ , \ \mathbf{s}_4=[s_{41},\ s_{42}]=\left[0,\ -A\sqrt{T}\right]$$

$$d_{12}=d_{13}=d_{14}=d_{24}=2A\sqrt{T/3}=2\sqrt{\varepsilon_{s_1}} \ , \ d_{34}=2A\sqrt{T}=\sqrt{2\varepsilon_{s_3}}$$

$$|\mathbf{s}_1|=|\mathbf{s}_2|=A\sqrt{T/3} \quad , \quad |\mathbf{s}_3|=|\mathbf{s}_4|=A\sqrt{T} \tag{1.4}$$

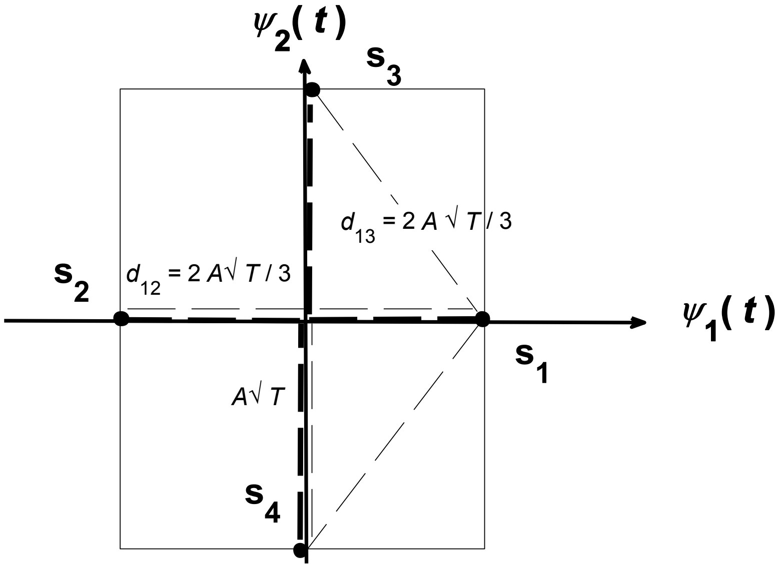


Fig. 1.3 Constellation diagram for the signal set, $s_1(t) \cdots s_4(t)$.

b. The block diagrams of demodulator as correlator and matched filter are given in Figs. 6.7a and 6.7b of Notes on Dimensionality of Signals_Sept 2012_HTE. They are not repeated here to save space.

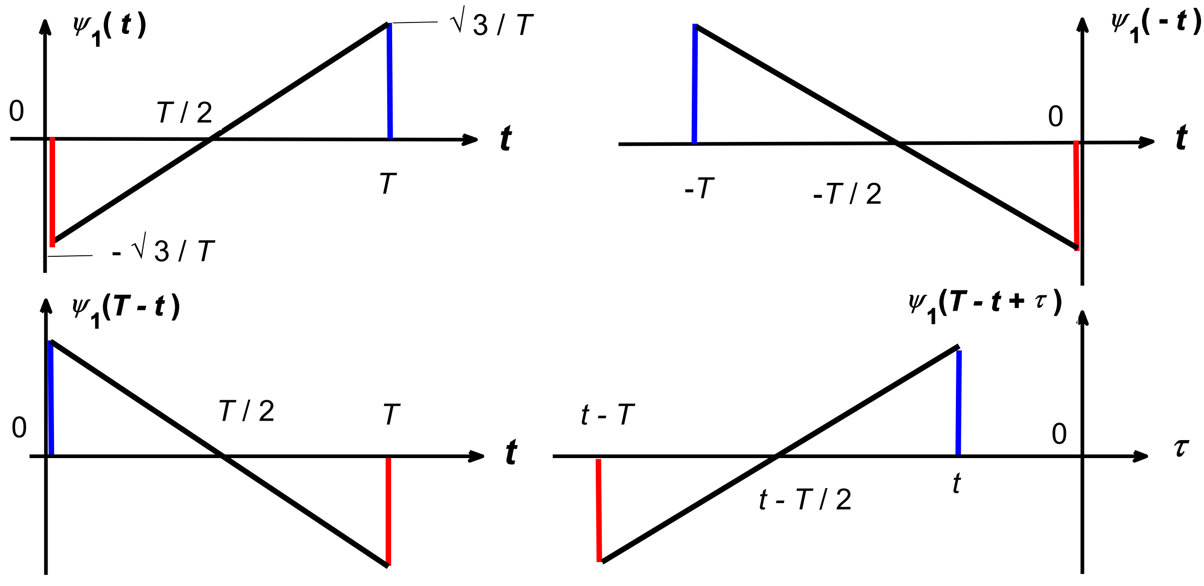


Fig. 1.4 Orientation of orthonormalized basis function, $\psi_1(t)$ for the convolution integral.

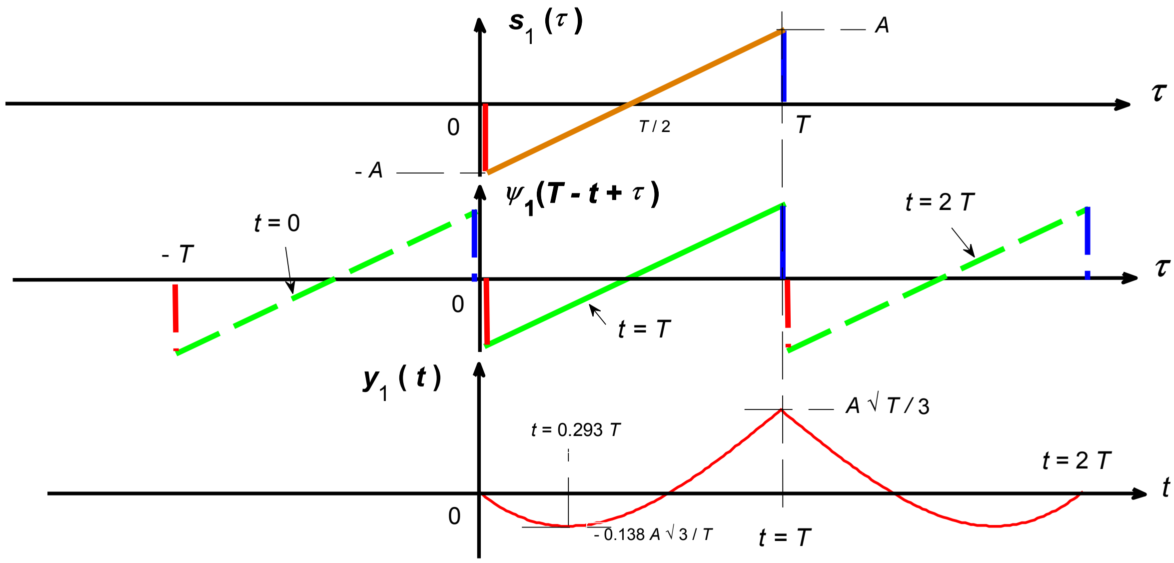


Fig. 1. 5 Graphical illustration of the convolution operation implemented in the upper matched filter of 4 QAM demodulator.

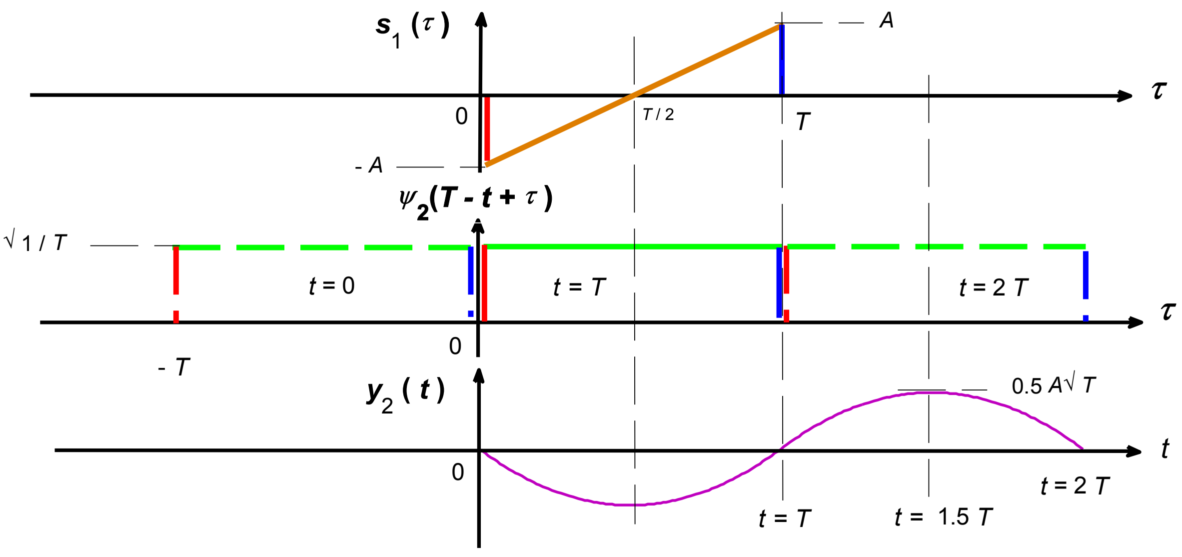


Fig. 1. 6 Graphical illustration of the convolution operation implemented in the lower matched filter of 4 QAM demodulator.

Taking into account (1.2), (1.4) and Fig. 1.4, we write

$$s(\tau)=A\left(\frac{2\tau}{T}-1\right)\ ,\ \psi_1(T-t+\tau)=\sqrt{3/T}\left[\frac{2(T-t+\tau)}{T}-1\right]\ ,\ \psi_2(T-t+\tau)=\sqrt{3/T} \qquad (1.5)$$

Then from Fig. 1.5, we get

$$y_1(t)=\begin{cases} y_{11}(t)=0 & t\le 0 \\ y_{12}(t)=\int_0^t s_1(\tau)\ \psi_1(T-t+\tau)\ d\tau = A\sqrt{3/T}\int_0^t\left(\frac{2\tau}{T}-1\right)\left[\frac{2(T-t+\tau)}{T}-1\right]d\tau & 0\le t\le T \\ y_{13}(t)=\int_{t-T}^{T} s_1(\tau)\ \psi_1(T-t+\tau)\ d\tau = A\sqrt{3/T}\int_{t-T}^{T}\left(\frac{2\tau}{T}-1\right)\left[\frac{2(T-t+\tau)}{T}-1\right]d\tau & T\le t\le 2T \\ y_{14}(t)=0 & t\ge 2T \end{cases} \qquad (1.6)$$

Performing the integrations with the help of m file named “ECE376_MT_2014_Q1”, we have

$$y_1(t)=\begin{cases} y_{12}(t)=A\sqrt{3/T}\left(-\frac{2t^3}{3T^2}+\frac{2t^2}{T}-t\right) & 0\le t\le T \\ y_{13}(t)=A\sqrt{3/T}\left(\frac{2t^3}{3T^2}-\frac{2t^2}{T}+t+\frac{2T}{3}\right) & T\le t\le 2T \end{cases} \qquad (1.7)$$

It is possible to verify that

$$
\begin{aligned}
&y_{12}(t=0)=0 \\
&y_{12}(t=T)=A\sqrt{T/3}=y_{13}(t=T) \\
&y_{13}(t=2T)=0
\end{aligned} \tag{1.8}
$$

$y_1(t)$ is plotted in Fig. 1.5. Doing the same for $y_2(t)$, we get

$$
y_2(t)=\begin{cases}
y_{21}(t)=0 & t\le 0 \\
y_{22}(t)=\int_0^t s_1(\tau)\ \psi_2(T-t+\tau)\ d\tau = A\sqrt{1/T}\int_0^t\left(\frac{2\tau}{T}-1\right)d\tau = A\sqrt{1/T}\left(\frac{t^2}{T}-t\right) & 0\le t\le T \\
y_{23}(t)=\int_{t-T}^T s_1(\tau)\ \psi_2(T-t+\tau)\ d\tau = A\sqrt{1/T}\int_{t-T}^T\left(\frac{2\tau}{T}-1\right)d\tau = A\sqrt{1/T}\left(-\frac{t^2}{T}+3t-2T\right) & T\le t\le 2T \\
y_{24}(t)=0 & t\ge 2T
\end{cases} \tag{1.9}
$$

From (1.9) and Fig. 1.6, we see that

$$
\begin{aligned}
&y_{22}(t=0)=0 \\
&y_{22}(t=T)=0=y_{23}(t=T) \\
&y_{23}(t=2T)=0
\end{aligned} \tag{1.10}
$$

c. At the sampling instance of $t=T$, from (1.8) and (1.10), we get

$$
\begin{aligned}
r_1 &= \int_0^T r(t)\ \psi_1(t)\ dt = \int_0^T s_1(t)\ \psi_1(t)\ dt + \int_0^T n(t)\ \psi_1(t)\ dt \\
&= A\sqrt{T/3}+n_1 \quad , \quad n_1=\int_0^T n(t)\ \psi_1(t)\ dt \\
r_2 &= \int_0^T r(t)\ \psi_2(t)\ dt = \int_0^T s_1(t)\ \psi_2(t)\ dt + \int_0^T n(t)\ \psi_2(t)\ dt \\
&= 0+n_2 \quad , \quad n_2=\int_0^T n(t)\ \psi_2(t)\ dt \quad , \quad \mathbf{r}=[r_1;\ r_2]=\begin{bmatrix} A\sqrt{T/3}+n_1 \\ n_2 \end{bmatrix}
\end{aligned} \tag{1.11}
$$

Carrying out the correlation metrics calculations, we will obtain the results stated in (1.12)

$$m=1\ ,\ C(\mathbf{r},\mathbf{s}_1)=2\,\mathbf{s}_1\cdot\mathbf{r}-\|\mathbf{s}_1\|^2=2\left[A\sqrt{T/3},\ 0\right]\begin{bmatrix}A\sqrt{T/3}+n_1\\ n_2\end{bmatrix}-A^2\frac{T}{3}=A^2\frac{T}{3}+2An_1\sqrt{\frac{T}{3}}$$

$$m=2\ ,\ C(\mathbf{r},\mathbf{s}_2)=2\,\mathbf{s}_2\cdot\mathbf{r}-\|\mathbf{s}_2\|^2=2\left[-A\sqrt{T/3},\ 0\right]\begin{bmatrix}A\sqrt{T/3}+n_1\\ n_2\end{bmatrix}-A^2\frac{T}{3}=-A^2T-2An_1\sqrt{\frac{T}{3}}$$

$$m=3\ ,\ C(\mathbf{r},\mathbf{s}_3)=2\,\mathbf{s}_3\cdot\mathbf{r}-\|\mathbf{s}_3\|^2=2\left[0,\ A\sqrt{T}\right]\begin{bmatrix}A\sqrt{T/3}+n_1\\ n_2\end{bmatrix}-A^2T=-A^2T+2An_2\sqrt{T}$$

$$m=4\ ,\ C(\mathbf{r},\mathbf{s}_4)=2\,\mathbf{s}_4\cdot\mathbf{r}-\|\mathbf{s}_4\|^2=2\left[0,\ -A\sqrt{T}\right]\begin{bmatrix}A\sqrt{T/3}+n_1\\ n_2\end{bmatrix}-A^2T=-A^2T-2An_2\sqrt{T} \quad (1.13)$$

Now for correct decision, the following three conditions must be satisfied

$$C(\mathbf{r},\mathbf{s}_1)>C(\mathbf{r},\mathbf{s}_2)\ :A^2\frac{T}{3}+2An_1\sqrt{\frac{T}{3}}>-A^2T-2An_1\sqrt{\frac{T}{3}}\ \rightarrow \overbrace{A\sqrt{T/3}}^{\sqrt{\varepsilon_p}}>-n_1\rightarrow r_1>0$$

$$C(\mathbf{r},\mathbf{s}_1)>C(\mathbf{r},\mathbf{s}_3)\ :\ \frac{A^2T}{3}+2An_1\sqrt{\frac{T}{3}}>-A^2T+2An_2\sqrt{T}\ \rightarrow\ \frac{1}{\sqrt{3}}\left(\overbrace{A\sqrt{T/3}+n_1}^{r_1}+A\sqrt{T/3}\right)>\overbrace{n_2}^{r_2}$$

$$\frac{1}{\sqrt{3}}\left(r_1+A\sqrt{T/3}\right)>r_2$$

$$C(\mathbf{r},\mathbf{s}_1)>C(\mathbf{r},\mathbf{s}_4)\ :\ \frac{A^2T}{3}+2An_1\sqrt{\frac{T}{3}}>-A^2T-2An_2\sqrt{T}\ \rightarrow\ \frac{1}{\sqrt{3}}\left(\overbrace{A\sqrt{T/3}+n_1}^{r_1}+A\sqrt{T/3}\right)>-\overbrace{n_2}^{r_2}$$

$$\frac{1}{\sqrt{3}}\left(r_1+A\sqrt{T/3}\right)>-r_2 \quad (1.13)$$

The decision regions can be plotted as

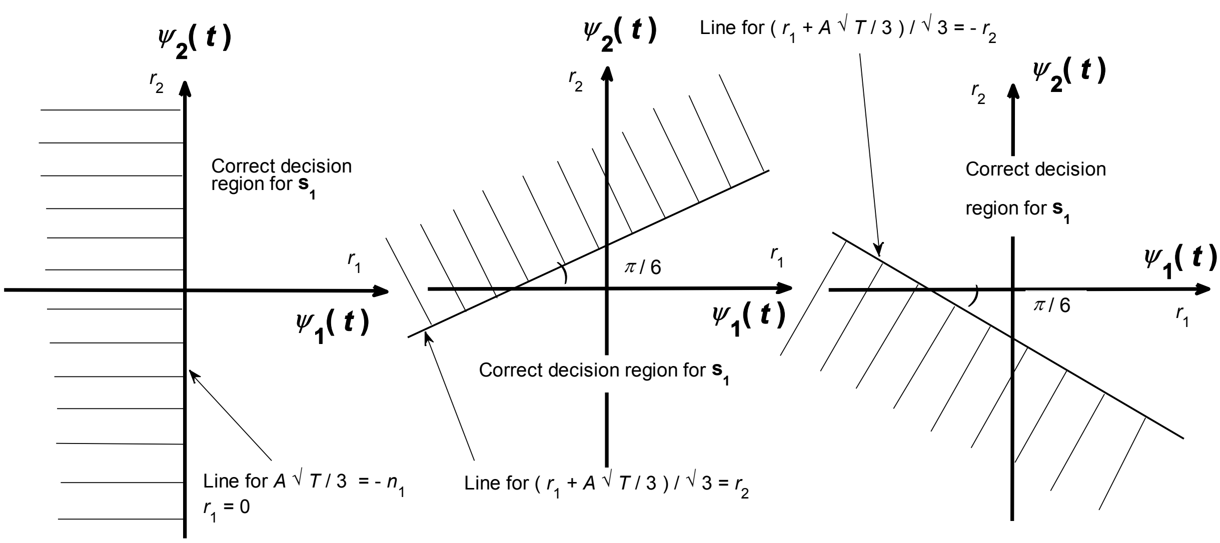


Fig. 1. 7 Decision regions for $\mathbf{s}_1$ .

Thus we write the probability of error for $\mathbf{s}_1$ in the following manner

$$P_e = 1 - P_c = 1 - \frac{1}{\left(\pi N_0\right)^{0.5}} \int_{-A\sqrt{T/3}}^{\infty} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \frac{1}{\left(\pi N_0\right)^{0.5}} \int_{-\frac{1}{\sqrt{3}}\left(2A\sqrt{T/3}+n_1\right)}^{\frac{1}{\sqrt{3}}\left(2A\sqrt{T/3}+n_1\right)} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \qquad (1.14)$$

2. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer

a) The role of the matched filter is to invert the input signal : False, the role of matched filter is to find the projection of the received signal onto respective orthogonal axes.

b) Under the conditions of equal energy, ASK has lower probability of error that PSK, QAM and FSK : False, for higher $M$, QAM is the best, then comes PSK, finally ASK. We have not made quantitative assessment on FSK.

c) In ASK, symbol duration is reduced as $M$ increases : False, independent of modulation type, symbol duration always increases as $M$ increases.

d) Optimum detector uses an optimum signal input : False, optimum detector takes into account statistical characteristics of noise, constellation diagram and attempts to make the optimum decision.

e) Correlation metrics evaluation makes a decision by measuring the distance between the received vector and all possible signals to be transmitted : False, correlation metrics evaluations give the correlations (similarities) between the received vector and the possibly transmitted signal vectors.

f) The number of correlation metrics evaluations depends on the dimensionality of the transmitted signal set : False, it depends on the $M$ value (number of levels).