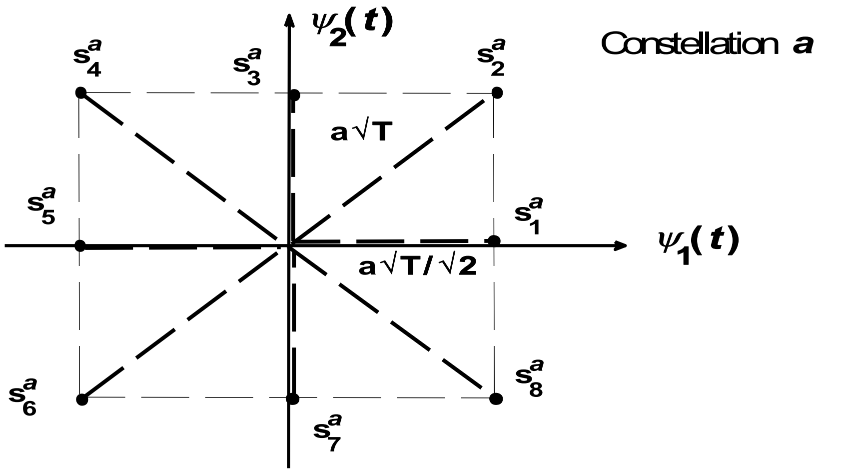
Çankaya University – ECE Department – ECE 588 (MT)

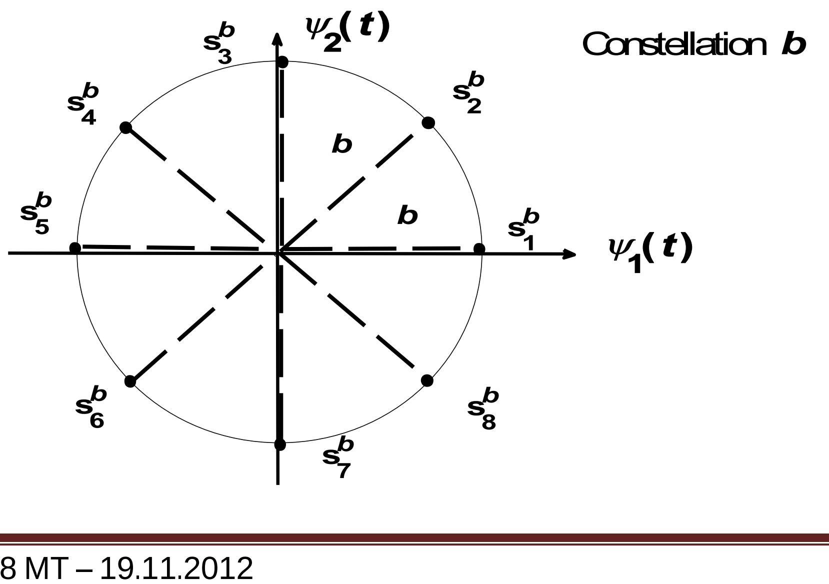
Student Name : Date : 19.11.2012
Student Number : Open Source Exam

Questions

1. (70 Points) Two constellations (a and b) are given as shown below.

   a. Find what the length of the signal vectors, b in the second constellation should be in terms of $a\sqrt{T}$ so that both constellations use the same total or average energy.
   b. Identify the type of modulation and dimensionality in these constellations. Write and make time waveform plots for $s_1^a(t)\cdots s_8^a(t)$ and $s_1^b(t)\cdots s_8^b(t)$, the signal vectors $\mathbf{s}_1^a\cdots\mathbf{s}_8^a$ and $\mathbf{s}_1^b\cdots\mathbf{s}_8^b$.
   c. Draw the demodulator as correlator and matched filter. Assuming that $s_4^a(t)$ from constellation a and $s_4^b(t)$ from constellation b were transmitted, find the outputs of the correlator and matched filter.
   d. Find the probability of error and decision regions via the evaluations of correlation metrics $C(\mathbf{r},\mathbf{s}_m)$ again assuming $s_4^a(t)$ from constellation a and $s_4^b(t)$ from constellation b were transmitted.





Solution : a. Provided that all signals in both constellations are sent from transmitter with equal probability, then

Total energy in constellation a is $4a^2T + 4\frac{a^2T}{2} = 6a^2T$, similarly

Total energy in constellation b is $8b^2$, setting $6a^2T = 8b^2$, we find $b = \frac{a}{2}\sqrt{3T}$

b. From the given constellations, it is easy to see that a is 8 QAM, whereas b is 8 PSK. The dimensionality in each case is two.

By adapting the following common orthonormalized basis functions,

$$\psi_1(t) = \begin{cases} \sqrt{2/T} & 0 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \sqrt{2/T} & T/2 \le t \le T \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

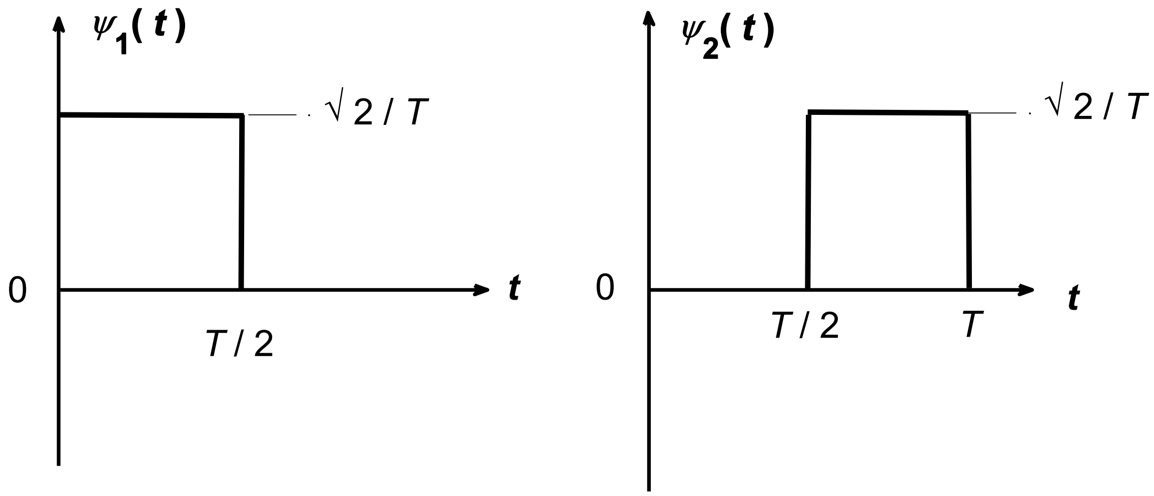


Fig. 1 The orthonormalized basis functions for Q1.

By using (1), Fig. 1 and Constellations a and b, we obtain the followings for the time waveforms of $s_1^a(t)\cdots s_8^a(t)$ and $s_1^b(t)\cdots s_8^b(t)$

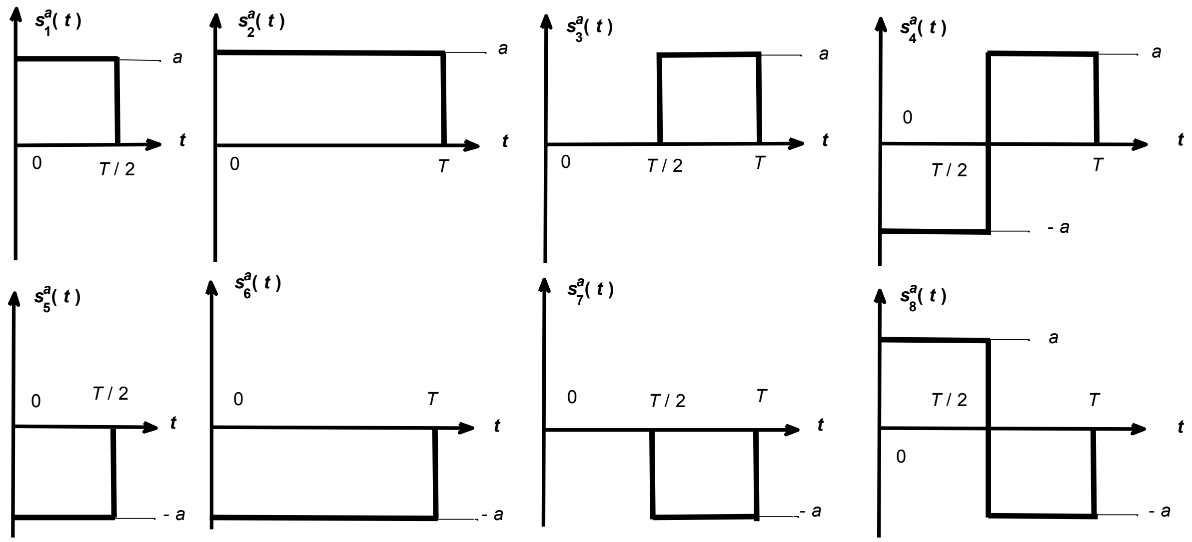


Fig. 2 Time waveforms of $s_1^a(t)\cdots s_8^a(t)$ for Q1.

From Fig. 2, it is possible to write the following expressions for $s_1^a(t)\cdots s_8^a(t)$

$$s_1^a(t)=\begin{cases} a & 0\le t\le T/2 \\ 0 & \text{otherwise}\end{cases},\ s_1^a(t)=a\sqrt{\frac{T}{2}}\psi_1(t)\ ,\ \mathbf{s_1^a}=\left[s_{11}^a,\ s_{12}^a\right]=\left[a\sqrt{\frac{T}{2}},\ 0\right],\ \varepsilon_{s_1^a}=\left\|\mathbf{s_1^a}\right\|^2=a^2\frac{T}{2}$$

$$s_2^a(t)=\begin{cases} a & 0\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_2^a(t)=a\sqrt{\frac{T}{2}}\psi_1(t)+a\sqrt{\frac{T}{2}}\psi_2(t)\ ,\ \mathbf{s_2^a}=\left[s_{21}^a,\ s_{22}^a\right]=\left[a\sqrt{\frac{T}{2}},\ a\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_2^a}=\left\|\mathbf{s_2^a}\right\|^2=a^2T$$

$$s_3^a(t)=\begin{cases} a & T/2\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_3^a(t)=a\sqrt{\frac{T}{2}}\psi_2(t)\ ,\ \mathbf{s_3^a}=\left[s_{31}^a,\ s_{32}^a\right]=\left[0,\ a\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_3^a}=a^2\frac{T}{2}$$

$$s_4^a(t)=\begin{cases} -a & 0\le t\le T/2 \\ a & T/2\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_4^a(t)=-a\sqrt{\frac{T}{2}}\psi_1(t)+a\sqrt{\frac{T}{2}}\psi_2(t)\ ,\ \mathbf{s_4^a}=\left[s_{41}^a,\ s_{42}^a\right]=\left[-a\sqrt{\frac{T}{2}},\ a\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_4^a}=a^2T$$

$$s_5^a(t)=\begin{cases} -a & 0\le t\le T/2 \\ 0 & \text{otherwise}\end{cases},\ s_5^a(t)=-a\sqrt{\frac{T}{2}}\psi_1(t)\ ,\ \mathbf{s_5^a}=\left[s_{51}^a,\ s_{52}^a\right]=\left[-a\sqrt{\frac{T}{2}},\ 0\right],\ \varepsilon_{s_5^a}=a^2\frac{T}{2}$$

$$s_6^a(t)=\begin{cases} -a & 0\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_6^a(t)=-a\sqrt{\frac{T}{2}}\psi_1(t)-a\sqrt{\frac{T}{2}}\psi_2(t)\ ,\ \mathbf{s_6^a}=\left[s_{61}^a,\ s_{62}^a\right]=\left[-a\sqrt{\frac{T}{2}},\ -a\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_6^a}=a^2T$$

$$s_7^a(t)=\begin{cases} -a & T/2\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_7^a(t)=-a\sqrt{\frac{T}{2}}\psi_2(t)\ ,\ \mathbf{s_7^a}=\left[s_{71}^a,\ s_{72}^a\right]=\left[0,\ -a\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_7^a}=a^2\frac{T}{2}$$

$$s_8^a(t)=\begin{cases} a & 0\le t\le T/2 \\ -a & T/2\le t\le T \\ 0 & \text{otherwise}\end{cases},\ s_8^a(t)=a\sqrt{\frac{T}{2}}\psi_1(t)-a\sqrt{\frac{T}{2}}\psi_2(t)\ ,\ \mathbf{s_8^a}=\left[s_{81}^a,\ s_{82}^a\right]=\left[a\sqrt{\frac{T}{2}},\ -a\sqrt{\frac{T}{2}}\right],\ \varepsilon_{s_8^a}=a^2T \quad (2)$$

The time waveforms of $s_1^b(t)\cdots s_8^b(t)$ are shown in Fig. 3.

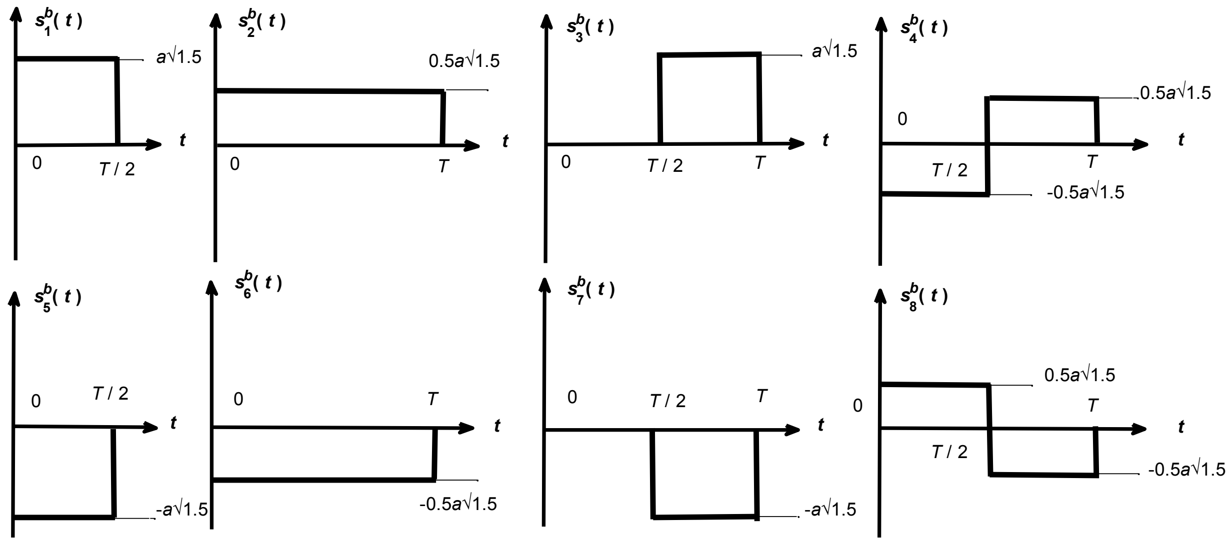


Fig. 3 Time waveforms of $s_1^b(t)\cdots s_8^b(t)$ for Q1.

From Fig. 3, it is possible to write the following expressions for $s_1^b(t)\cdots s_8^b(t)$

$$s_1^b(t)=\begin{cases} a\sqrt{\dfrac{3}{2}}=b\sqrt{\dfrac{2}{T}} & 0\le t\le T/2 \\ 0 & \text{otherwise} \end{cases}\ ,\ s_1^b(t)=\frac{a\sqrt{3T}}{2}\psi_1(t)=b\psi_1(t)$$

$$\mathbf{s_1^b}=\left[s_{11}^b,\ s_{12}^b\right]=\left[\frac{a\sqrt{3T}}{2},\ 0\right]=\left[b,\ 0\right]\ ,\ \varepsilon_{s_1^b}=\left\|\mathbf{s_1^b}\right\|^2=\frac{3a^2T}{4}=b^2$$

$$s_2^b(t)=\begin{cases} a\dfrac{\sqrt{3}}{2}=\dfrac{b}{\sqrt{T}} & 0\le t\le T \\ 0 & \text{otherwise} \end{cases}\ ,\ s_2^b(t)=\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_1(t)+\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_2(t)=\frac{b}{\sqrt{2}}\psi_1(t)+\frac{b}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_2^b}=\left[s_{21}^b,\ s_{22}^b\right]=\left[\frac{a}{2}\sqrt{\frac{3T}{2}},\ \frac{a}{2}\sqrt{\frac{3T}{2}}\right]=\left[\frac{b}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right]\ ,\ \varepsilon_{s_2^b}=\left\|\mathbf{s_2^b}\right\|^2=\frac{3a^2T}{4}=b^2$$

$$s_3^b(t)=\begin{cases} a\sqrt{\dfrac{3}{2}}=b\sqrt{\dfrac{2}{T}} & T/2\le t\le T \\ 0 & \text{otherwise} \end{cases}\ ,\ s_1^b(t)=\frac{a\sqrt{3T}}{2}\psi_2(t)=b\psi_2(t)$$

$$\mathbf{s_3^b}=\left[s_{31}^b,\ s_{32}^b\right]=\left[0,\ \frac{a\sqrt{3T}}{2}\right]=\left[0,\ b\right]\ ,\ \varepsilon_{s_1^b}=\frac{3a^2T}{4}=b^2$$

$$s_4^b(t)=\begin{cases}-a\dfrac{\sqrt{3}}{2}=-\dfrac{b}{\sqrt{T}} & 0\le t\le T/2\\ a\dfrac{\sqrt{3}}{2}=\dfrac{b}{\sqrt{T}} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\; s_4^b(t)=-\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_1(t)+\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_2(t)=-\frac{b}{\sqrt{2}}\psi_1(t)+\frac{b}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_4^b}=\left[s_{41}^b,\ s_{42}^b\right]=\left[-\frac{a}{2}\sqrt{\frac{3T}{2}},\ \frac{a}{2}\sqrt{\frac{3T}{2}}\right]=\left[-\frac{b}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right],\ \varepsilon_{s_4^b}=\frac{3a^2T}{4}=b^2$$

$$s_5^b(t)=\begin{cases}-a\sqrt{\dfrac{3}{2}}=-b\sqrt{\dfrac{2}{T}} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases},\; s_5^b(t)=-\frac{a\sqrt{3T}}{2}\psi_1(t)=-b\psi_1(t)$$

$$\mathbf{s_5^b}=\left[s_{51}^b,\ s_{52}^b\right]=\left[-\frac{a\sqrt{3T}}{2},\ 0\right]=\left[-b,\ 0\right],\ \varepsilon_{s_5^b}=\frac{3a^2T}{4}=b^2$$

$$s_6^b(t)=\begin{cases}-a\dfrac{\sqrt{3}}{2}=-\dfrac{b}{\sqrt{T}} & 0\le t\le T\\ 0 & \text{otherwise}\end{cases},\; s_6^b(t)=-\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_1(t)-\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_2(t)=-\frac{b}{\sqrt{2}}\psi_1(t)-\frac{b}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_6^b}=\left[s_{61}^b,\ s_{62}^b\right]=\left[-\frac{a}{2}\sqrt{\frac{3T}{2}},\ -\frac{a}{2}\sqrt{\frac{3T}{2}}\right]=\left[-\frac{b}{\sqrt{2}},\ -\frac{b}{\sqrt{2}}\right],\ \varepsilon_{s_6^b}=\frac{3a^2T}{4}=b^2$$

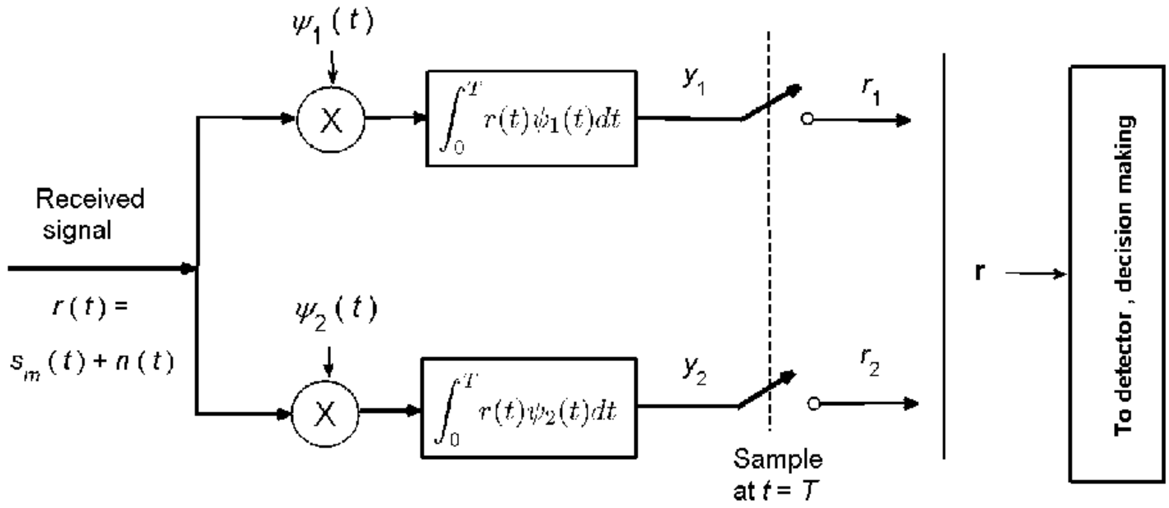
$$s_7^b(t)=\begin{cases}-a\sqrt{\dfrac{3}{2}}=-b\sqrt{\dfrac{2}{T}} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\; s_1^b(t)=-\frac{a\sqrt{3T}}{2}\psi_2(t)=-b\psi_2(t)$$

$$\mathbf{s_7^b}=\left[s_{71}^b,\ s_{72}^b\right]=\left[0,\ -\frac{a\sqrt{3T}}{2}\right]=\left[0,\ -b\right],\ \varepsilon_{s_7^b}=\frac{3a^2T}{4}=b^2$$

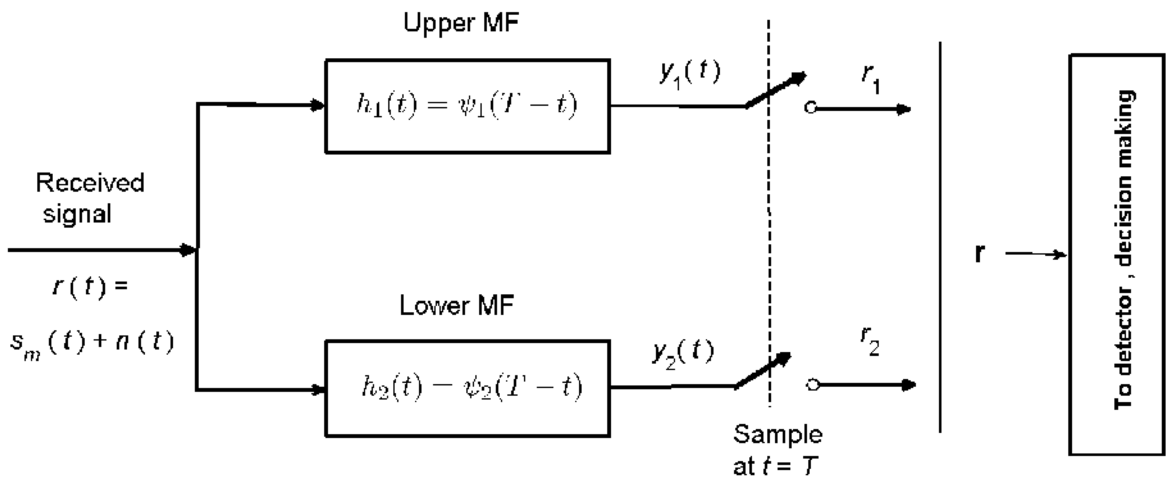
$$s_8^b(t)=\begin{cases}a\dfrac{\sqrt{3}}{2}=\dfrac{b}{\sqrt{T}} & 0\le t\le T/2\\ -a\dfrac{\sqrt{3}}{2}=-\dfrac{b}{\sqrt{T}} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\; s_8^b(t)=\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_1(t)-\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_2(t)=\frac{b}{\sqrt{2}}\psi_1(t)-\frac{b}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_8^b}=\left[s_{81}^b,\ s_{82}^b\right]=\left[\frac{a}{2}\sqrt{\frac{3T}{2}},\ -\frac{a}{2}\sqrt{\frac{3T}{2}}\right]=\left[\frac{b}{\sqrt{2}},\ -\frac{b}{\sqrt{2}}\right],\ \varepsilon_{s_8^b}=\frac{3a^2T}{4}=b^2 \qquad (3)$$

c. Since QAM and PSK are both two dimensional, for block diagrams of correlator and MF, we benefit from Fig. 6.7 of Notes on Dimensionality of Signals_Sept 2012_HTE, which we reproduce below in Fig. 4



a) Block diagram of correlator type of demodulator.



b) Block diagram of matched filter (MF) type of demodulator.

Fig. 4 Block diagrams of correlator and matched filter type of demodulators for the signal set in Q1.

If $s_4^a(t)$ is sent from the transmitter in Constellation a and $s_4^b(t)$ is sent from the transmitter in Constellation b, then the output from the upper and lower braches of the correlator will be (before or after sampling)

$$y_1^a=\int_0^T r(t)\ \psi_1(t)dt=\int_0^T s_4^a(t)\ \psi_1(t)dt+\int_0^T n(t)\ \psi_1(t)dt=s_{41}^a+n_1=-a\sqrt{\frac{T}{2}}+n_1$$
$$y_2^a=\int_0^T r(t)\ \psi_2(t)dt=\int_0^T s_4^a(t)\ \psi_2(t)dt+\int_0^T n(t)\ \psi_2(t)dt=s_{42}^a+n_2=a\sqrt{\frac{T}{2}}+n_2$$
$$y_1^b=\int_0^T r(t)\ \psi_1(t)dt=\int_0^T s_4^b(t)\ \psi_1(t)dt+\int_0^T n(t)\ \psi_1(t)dt=s_{41}^b+n_1=-\frac{a}{2}\sqrt{\frac{3T}{2}}+n_1=-\frac{b}{\sqrt{2}}+n_1$$
$$y_2^b=\int_0^T r(t)\ \psi_2(t)dt=\int_0^T s_4^b(t)\ \psi_2(t)dt+\int_0^T n(t)\ \psi_2(t)dt=s_{42}^b+n_2=\frac{a}{2}\sqrt{\frac{3T}{2}}+n_2=\frac{b}{\sqrt{2}}+n_2 \qquad (4)$$

We know that output from MF will be identical to (4) at the time of sampling at $t=T$ which means that we can construct the received vector $\mathbf{r}$ that we supply to the detector and which will be used in the decision making process, as follows

$$\mathbf{r^a}=\begin{bmatrix} r_1^a \\ r_2^a \end{bmatrix}=\begin{bmatrix} -a\sqrt{\frac{T}{2}}+n_1 \\ a\sqrt{\frac{T}{2}}+n_2 \end{bmatrix} \quad , \quad \mathbf{r^b}=\begin{bmatrix} r_1^b \\ r_2^b \end{bmatrix}=\begin{bmatrix} -\frac{a}{2}\sqrt{\frac{3T}{2}}+n_1 \\ \frac{a}{2}\sqrt{\frac{3T}{2}}+n_2 \end{bmatrix}=\begin{bmatrix} -\frac{b}{\sqrt{2}}+n_1 \\ \frac{b}{\sqrt{2}}+n_2 \end{bmatrix} \qquad (5)$$

d. Using (5), we evaluate correlation metrics $C(\mathbf{r},\mathbf{s}_m)$ for $m=1\cdots 8$ as follows

$$C(\mathbf{r^a},\mathbf{s_m^a})=2\,\mathbf{s_m^a}\cdot\mathbf{r^a}-\|\mathbf{s_m^a}\|^2 \ , \quad m=1\cdots 8$$

$$m=1\ ,\ C(\mathbf{r^a},\mathbf{s_1^a})=2\,\mathbf{s_1^a}\cdot\mathbf{r^a}-\|\mathbf{s_1^a}\|^2=2\begin{bmatrix} a\sqrt{\frac{T}{2}}, & 0\end{bmatrix}\begin{bmatrix} -a\sqrt{\frac{T}{2}}+n_1 \\ a\sqrt{\frac{T}{2}}+n_2 \end{bmatrix}-a^2\frac{T}{2}=an_1\sqrt{2T}-\frac{3a^2T}{2}$$

$$m=2\ ,\ C(\mathbf{r^a},\mathbf{s_2^a})=2\,\mathbf{s_2^a}\cdot\mathbf{r^a}-\|\mathbf{s_2^a}\|^2=2\begin{bmatrix} a\sqrt{\frac{T}{2}}, & a\sqrt{\frac{T}{2}}\end{bmatrix}\begin{bmatrix} -a\sqrt{\frac{T}{2}}+n_1 \\ a\sqrt{\frac{T}{2}}+n_2 \end{bmatrix}-a^2T=an_1\sqrt{2T}+an_2\sqrt{2T}-a^2T$$

$$m=3\ ,\ C(\mathbf{r^a},\mathbf{s_3^a})=2\,\mathbf{s_3^a}\cdot\mathbf{r^a}-\|\mathbf{s_3^a}\|^2=2\begin{bmatrix} 0, & a\sqrt{\frac{T}{2}}\end{bmatrix}\begin{bmatrix} -a\sqrt{\frac{T}{2}}+n_1 \\ a\sqrt{\frac{T}{2}}+n_2 \end{bmatrix}-a^2\frac{T}{2}=an_2\sqrt{2T}+\frac{a^2T}{2}$$

$$m=4\ ,\ C(\mathbf{r^a},\mathbf{s_4^a})=2\,\mathbf{s_4^a}\cdot\mathbf{r^a}-\|\mathbf{s_4^a}\|^2=2\begin{bmatrix} -a\sqrt{\frac{T}{2}}, & a\sqrt{\frac{T}{2}}\end{bmatrix}\begin{bmatrix} -a\sqrt{\frac{T}{2}}+n_1 \\ a\sqrt{\frac{T}{2}}+n_2 \end{bmatrix}-a^2T=an_2\sqrt{2T}-an_1\sqrt{2T}+a^2T$$

$$m=5\ ,\ C\left(\mathbf{r}^{a},\mathbf{s}_5^{a}\right)=2\,\mathbf{s}_5^{a}\cdot\mathbf{r}^{a}-\left\|\mathbf{s}_5^{a}\right\|^2=2\left[-a\sqrt{\frac{T}{2}},\ 0\right]\begin{bmatrix}-a\sqrt{\frac{T}{2}}+n_1\\ a\sqrt{\frac{T}{2}}+n_2\end{bmatrix}-a^2\frac{T}{2}=-an_1\sqrt{2T}+\frac{a^2T}{2}$$

$$m=6\ ,\ C\left(\mathbf{r}^{a},\mathbf{s}_6^{a}\right)=2\,\mathbf{s}_6^{a}\cdot\mathbf{r}^{a}-\left\|\mathbf{s}_6^{a}\right\|^2=2\left[-a\sqrt{\frac{T}{2}},\ -a\sqrt{\frac{T}{2}}\right]\begin{bmatrix}-a\sqrt{\frac{T}{2}}+n_1\\ a\sqrt{\frac{T}{2}}+n_2\end{bmatrix}-a^2T=-an_1\sqrt{2T}-an_2\sqrt{2T}-a^2T$$

$$m=7\ ,\ C\left(\mathbf{r}^{a},\mathbf{s}_7^{a}\right)=2\,\mathbf{s}_7^{a}\cdot\mathbf{r}^{a}-\left\|\mathbf{s}_7^{a}\right\|^2=2\left[0,\ -a\sqrt{\frac{T}{2}}\right]\begin{bmatrix}-a\sqrt{\frac{T}{2}}+n_1\\ a\sqrt{\frac{T}{2}}+n_2\end{bmatrix}-a^2\frac{T}{2}=-an_2\sqrt{2T}-\frac{3a^2T}{2}$$

$$m=8\ ,\ C\left(\mathbf{r}^{a},\mathbf{s}_8^{a}\right)=2\,\mathbf{s}_8^{a}\cdot\mathbf{r}^{a}-\left\|\mathbf{s}_8^{a}\right\|^2=2\left[a\sqrt{\frac{T}{2}},\ -a\sqrt{\frac{T}{2}}\right]\begin{bmatrix}-a\sqrt{\frac{T}{2}}+n_1\\ a\sqrt{\frac{T}{2}}+n_2\end{bmatrix}-a^2T=an_1\sqrt{2T}-an_2\sqrt{2T}-3a^2T \quad (6)$$

In Costellation a, the correct decision region for $\mathbf{s}_4^{a}$ for the case of $s_4^a(t)$ being transmitted is defined by the following inequalities and corresponding conditions.

$$C\left(\mathbf{r}^{a},\mathbf{s}_4^{a}\right)>C\left(\mathbf{r}^{a},\mathbf{s}_1^{a}\right):an_2\sqrt{2T}-an_1\sqrt{2T}+a^2T>an_1\sqrt{2T}-\frac{3a^2T}{2}\ \rightarrow 1.25a\sqrt{\frac{T}{2}}+\frac{n_2}{2}>n_1$$

$$C\left(\mathbf{r}^{a},\mathbf{s}_4^{a}\right)>C\left(\mathbf{r}^{a},\mathbf{s}_2^{a}\right):an_2\sqrt{2T}-an_1\sqrt{2T}+a^2T>an_1\sqrt{2T}+an_2\sqrt{2T}-a^2T\ \rightarrow\ a\sqrt{\frac{T}{2}}>n_1$$

$$C\left(\mathbf{r}^{a},\mathbf{s}_4^{a}\right)>C\left(\mathbf{r}^{a},\mathbf{s}_3^{a}\right):an_2\sqrt{2T}-an_1\sqrt{2T}+a^2T>an_2\sqrt{2T}+\frac{a^2T}{2}\ \rightarrow\frac{a}{2}\sqrt{\frac{T}{2}}>n_1$$

$$C\left(\mathbf{r}^{a},\mathbf{s}_4^{a}\right)>C\left(\mathbf{r}^{a},\mathbf{s}_5^{a}\right):an_2\sqrt{2T}-an_1\sqrt{2T}+a^2T>-an_1\sqrt{2T}+\frac{a^2T}{2}\ \rightarrow\frac{a}{2}\sqrt{\frac{T}{2}}>-n_2$$

$$C\left(\mathbf{r}^{a},\mathbf{s}_4^{a}\right)>C\left(\mathbf{r}^{a},\mathbf{s}_6^{a}\right):an_2\sqrt{2T}-an_1\sqrt{2T}+a^2T>-an_1\sqrt{2T}-an_2\sqrt{2T}-a^2T\ \rightarrow a\sqrt{\frac{T}{2}}>-n_2$$

$$C\left(\mathbf{r}^{a},\mathbf{s}_4^{a}\right)>C\left(\mathbf{r}^{a},\mathbf{s}_7^{a}\right):an_2\sqrt{2T}-an_1\sqrt{2T}+a^2T>-an_2\sqrt{2T}-\frac{3a^2T}{2}\ \rightarrow 2.5a\sqrt{\frac{T}{2}}-2n_2>n_1$$

$$C\left(\mathbf{r}^{a},\mathbf{s}_4^{a}\right)>C\left(\mathbf{r}^{a},\mathbf{s}_8^{a}\right):an_2\sqrt{2T}-an_1\sqrt{2T}+a^2T>an_1\sqrt{2T}-an_2\sqrt{2T}-3a^2T\ \rightarrow 2a\sqrt{\frac{T}{2}}+n_2>n_1 \quad (7)$$

Repeating the same for Constellation b, we get

$$C\left(\mathbf{r}^{\mathbf{b}},\, \mathbf{s}_{\mathbf{m}}^{\mathbf{b}}\right) = 2\, \mathbf{s}_{\mathbf{m}}^{\mathbf{b}} \cdot \mathbf{r}^{\mathbf{b}} - \left\| \mathbf{s}_{\mathbf{m}}^{\mathbf{b}} \right\|^2 \quad , \quad m = 1 \cdots 8$$

$$m = 1\ ,\ C\left(\mathbf{r}^{\mathbf{b}},\, \mathbf{s}_{1}^{\mathbf{b}}\right) = 2\, \mathbf{s}_{1}^{\mathbf{b}} \cdot \mathbf{r}^{\mathbf{b}} - \left\| \mathbf{s}_{1}^{\mathbf{b}} \right\|^2 = 2\left[\frac{a\sqrt{3T}}{2},\ 0\right]\begin{bmatrix} -\dfrac{a}{2}\sqrt{\dfrac{3T}{2}} + n_1 \\ \dfrac{a}{2}\sqrt{\dfrac{3T}{2}} + n_2 \end{bmatrix} - \frac{3a^2T}{4} = an_1\sqrt{3T} - \frac{3a^2T}{4}\left(\sqrt{2}+1\right)$$

$$= 2\left[b,\ 0\right]\begin{bmatrix} -\dfrac{b}{\sqrt{2}} + n_1 \\ \dfrac{b}{\sqrt{2}} + n_2 \end{bmatrix} - b^2 = 2bn_1 - b^2\left(\sqrt{2}+1\right)$$

$$m = 2\ ,\ C\left(\mathbf{r}^{\mathbf{b}},\, \mathbf{s}_{2}^{\mathbf{b}}\right) = 2\, \mathbf{s}_{2}^{\mathbf{b}} \cdot \mathbf{r}^{\mathbf{b}} - \left\| \mathbf{s}_{2}^{\mathbf{b}} \right\|^2 = 2\left[\frac{a}{2}\sqrt{\frac{3T}{2}},\ \frac{a}{2}\sqrt{\frac{3T}{2}}\right]\begin{bmatrix} -\dfrac{a}{2}\sqrt{\dfrac{3T}{2}} + n_1 \\ \dfrac{a}{2}\sqrt{\dfrac{3T}{2}} + n_2 \end{bmatrix} - \frac{3a^2T}{4} = an_1\sqrt{\frac{3T}{2}} + an_2\sqrt{\frac{3T}{2}} - \frac{3a^2T}{4}$$

$$= 2\left[\frac{b}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right]\begin{bmatrix} -\dfrac{b}{\sqrt{2}} + n_1 \\ \dfrac{b}{\sqrt{2}} + n_2 \end{bmatrix} - b^2 = bn_1\sqrt{2} + bn_2\sqrt{2} - b^2$$

$$m = 3\ ,\ C\left(\mathbf{r}^{\mathbf{b}},\, \mathbf{s}_{3}^{\mathbf{b}}\right) = 2\, \mathbf{s}_{3}^{\mathbf{b}} \cdot \mathbf{r}^{\mathbf{b}} - \left\| \mathbf{s}_{3}^{\mathbf{b}} \right\|^2 = 2\left[0,\ \frac{a\sqrt{3T}}{2}\right]\begin{bmatrix} -\dfrac{a}{2}\sqrt{\dfrac{3T}{2}} + n_1 \\ \dfrac{a}{2}\sqrt{\dfrac{3T}{2}} + n_2 \end{bmatrix} - \frac{3a^2T}{4} = an_2\sqrt{3T} + \frac{3a^2T}{4}\left(\sqrt{2}-1\right)$$

$$= 2\left[0,\ b\right]\begin{bmatrix} -\dfrac{b}{\sqrt{2}} + n_1 \\ \dfrac{b}{\sqrt{2}} + n_2 \end{bmatrix} - b^2 = 2bn_2 + b^2\left(\sqrt{2}-1\right)$$

$$m = 4\ ,\ C\left(\mathbf{r}^{\mathbf{b}},\, \mathbf{s}_{4}^{\mathbf{b}}\right) = 2\, \mathbf{s}_{4}^{\mathbf{b}} \cdot \mathbf{r}^{\mathbf{b}} - \left\| \mathbf{s}_{4}^{\mathbf{b}} \right\|^2 = 2\left[-\frac{a}{2}\sqrt{\frac{3T}{2}},\ \frac{a}{2}\sqrt{\frac{3T}{2}}\right]\begin{bmatrix} -\dfrac{a}{2}\sqrt{\dfrac{3T}{2}} + n_1 \\ \dfrac{a}{2}\sqrt{\dfrac{3T}{2}} + n_2 \end{bmatrix} - \frac{3a^2T}{4} = an_2\sqrt{\frac{3T}{2}} - an_1\sqrt{\frac{3T}{2}} + \frac{3a^2T}{4}$$

$$= 2\left[\frac{-b}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right]\begin{bmatrix} -\dfrac{b}{\sqrt{2}} + n_1 \\ \dfrac{b}{\sqrt{2}} + n_2 \end{bmatrix} - b^2 = bn_2\sqrt{2} - bn_1\sqrt{2} + b^2$$

$$m=5\ ,\ C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_5^{\mathrm{b}}\right)=2\,\mathbf{s}_5^{\mathrm{b}}\cdot\mathbf{r}^{\mathrm{b}}-\left\|\mathbf{s}_5^{\mathrm{b}}\right\|^2=2\left[-\frac{a\sqrt{3T}}{2},\ 0\right]\begin{bmatrix}-\frac{a}{2}\sqrt{\frac{3T}{2}}+n_1\\ \frac{a}{2}\sqrt{\frac{3T}{2}}+n_2\end{bmatrix}-\frac{3a^2T}{4}=-an_1\sqrt{3T}+\frac{3a^2T}{4}\left(\sqrt{2}-1\right)$$

$$=2\left[-b,\ 0\right]\begin{bmatrix}-\frac{b}{\sqrt{2}}+n_1\\ \frac{b}{\sqrt{2}}+n_2\end{bmatrix}-b^2=-2bn_1+b^2\left(\sqrt{2}-1\right)$$

$$m=6\ ,\ C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_6^{\mathrm{b}}\right)=2\,\mathbf{s}_6^{\mathrm{b}}\cdot\mathbf{r}^{\mathrm{b}}-\left\|\mathbf{s}_6^{\mathrm{b}}\right\|^2=2\left[-\frac{a}{2}\sqrt{\frac{3T}{2}},\ -\frac{a}{2}\sqrt{\frac{3T}{2}}\right]\begin{bmatrix}-\frac{a}{2}\sqrt{\frac{3T}{2}}+n_1\\ \frac{a}{2}\sqrt{\frac{3T}{2}}+n_2\end{bmatrix}-\frac{3a^2T}{4}=-an_1\sqrt{\frac{3T}{2}}-an_2\sqrt{\frac{3T}{2}}-\frac{3a^2T}{4}$$

$$=2\left[\frac{-b}{\sqrt{2}},\ \frac{-b}{\sqrt{2}}\right]\begin{bmatrix}-\frac{b}{\sqrt{2}}+n_1\\ \frac{b}{\sqrt{2}}+n_2\end{bmatrix}-b^2=-bn_1\sqrt{2}-bn_2\sqrt{2}-b^2$$

$$m=7\ ,\ C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_7^{\mathrm{b}}\right)=2\,\mathbf{s}_7^{\mathrm{b}}\cdot\mathbf{r}^{\mathrm{b}}-\left\|\mathbf{s}_7^{\mathrm{b}}\right\|^2=2\left[0,\ -\frac{a\sqrt{3T}}{2}\right]\begin{bmatrix}-\frac{a}{2}\sqrt{\frac{3T}{2}}+n_1\\ \frac{a}{2}\sqrt{\frac{3T}{2}}+n_2\end{bmatrix}-\frac{3a^2T}{4}=-an_2\sqrt{3T}-\frac{3a^2T}{4}\left(\sqrt{2}+1\right)$$

$$=2\left[0,\ -b\right]\begin{bmatrix}-\frac{b}{\sqrt{2}}+n_1\\ \frac{b}{\sqrt{2}}+n_2\end{bmatrix}-b^2=-2bn_2-b^2\left(\sqrt{2}+1\right)$$

$$m=8\ ,\ C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_8^{\mathrm{b}}\right)=2\,\mathbf{s}_8^{\mathrm{b}}\cdot\mathbf{r}^{\mathrm{b}}-\left\|\mathbf{s}_8^{\mathrm{b}}\right\|^2=2\left[\frac{a}{2}\sqrt{\frac{3T}{2}},\ -\frac{a}{2}\sqrt{\frac{3T}{2}}\right]\begin{bmatrix}-\frac{a}{2}\sqrt{\frac{3T}{2}}+n_1\\ \frac{a}{2}\sqrt{\frac{3T}{2}}+n_2\end{bmatrix}-\frac{3a^2T}{4}=an_1\sqrt{\frac{3T}{2}}-an_2\sqrt{\frac{3T}{2}}-\frac{9a^2T}{4}$$

$$=2\left[\frac{b}{\sqrt{2}},\ \frac{-b}{\sqrt{2}}\right]\begin{bmatrix}-\frac{b}{\sqrt{2}}+n_1\\ \frac{b}{\sqrt{2}}+n_2\end{bmatrix}-b^2=bn_1\sqrt{2}-bn_2\sqrt{2}-3b^2$$

$$C(\mathbf{r}^b,\mathbf{s}_4^b)>C(\mathbf{r}^b,\mathbf{s}_1^b)\ : an_2\sqrt{\frac{3T}{2}}-an_1\sqrt{\frac{3T}{2}}+\frac{3a^2T}{4}>an_1\sqrt{3T}-\frac{3a^2T}{4}\left(\sqrt{2}+1\right)\ \to\frac{a}{2}\sqrt{3T}+\frac{n_2}{1+\sqrt{2}}>n_1$$

$$:bn_2\sqrt{2}-bn_1\sqrt{2}+b^2>2bn_1-b^2\left(\sqrt{2}+1\right)\ \to b+\frac{n_2}{1+\sqrt{2}}>n_1$$

$$C(\mathbf{r}^b,\mathbf{s}_4^b)>C(\mathbf{r}^b,\mathbf{s}_2^b)\ : an_2\sqrt{\frac{3T}{2}}-an_1\sqrt{\frac{3T}{2}}+\frac{3a^2T}{4}>an_1\sqrt{\frac{3T}{2}}+an_2\sqrt{\frac{3T}{2}}-\frac{3a^2T}{4}\ \to\ \frac{a}{2}\sqrt{\frac{3T}{2}}>n_1$$

$$:bn_2\sqrt{2}-bn_1\sqrt{2}+b^2>bn_1\sqrt{2}+bn_2\sqrt{2}-b^2\ \to\ \frac{b}{\sqrt{2}}>n_1$$

$$C(\mathbf{r}^b,\mathbf{s}_4^b)>C(\mathbf{r}^b,\mathbf{s}_3^b)\ : an_2\sqrt{\frac{3T}{2}}-an_1\sqrt{\frac{3T}{2}}+\frac{3a^2T}{4}>an_2\sqrt{3T}+\frac{3a^2T}{4}\left(\sqrt{2}-1\right)\ \to a\sqrt{\frac{3T}{2}}\left(1-\frac{1}{\sqrt{2}}\right)+n_2\left(1-\sqrt{2}\right)>n_1$$

$$:bn_2\sqrt{2}-bn_1\sqrt{2}+b^2>2bn_2+b^2\left(\sqrt{2}-1\right)\ \to b\left(\sqrt{2}-1\right)+n_2\left(1-\sqrt{2}\right)>n_1$$

$$C(\mathbf{r}^b,\mathbf{s}_4^b)>C(\mathbf{r}^b,\mathbf{s}_5^b)\ : an_2\sqrt{\frac{3T}{2}}-an_1\sqrt{\frac{3T}{2}}+\frac{3a^2T}{4}>-an_1\sqrt{3T}+\frac{3a^2T}{4}\left(\sqrt{2}-1\right)\ \to -\frac{a}{2}\sqrt{3T}-\frac{n_2}{\sqrt{2}-1}>n_1$$

$$:bn_2\sqrt{2}-bn_1\sqrt{2}+b^2>-2bn_1+b^2\left(\sqrt{2}-1\right)\ \to -b-\frac{n_2}{\sqrt{2}-1}>n_1$$

$$C(\mathbf{r}^b,\mathbf{s}_4^b)>C(\mathbf{r}^b,\mathbf{s}_6^b)\ : an_2\sqrt{\frac{3T}{2}}-an_1\sqrt{\frac{3T}{2}}+\frac{3a^2T}{4}>-an_1\sqrt{\frac{3T}{2}}-an_2\sqrt{\frac{3T}{2}}-\frac{3a^2T}{4}\ \to\frac{a}{2}\sqrt{\frac{3T}{2}}>-n_2$$

$$:bn_2\sqrt{2}-bn_1\sqrt{2}+b^2>-bn_1\sqrt{2}-bn_2\sqrt{2}-b^2\ \to\frac{b}{\sqrt{2}}>-n_2$$

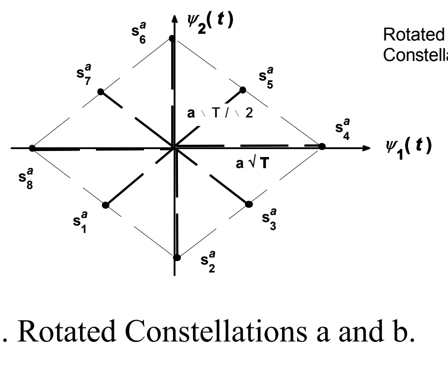
$$C(\mathbf{r}^b,\mathbf{s}_4^b)>C(\mathbf{r}^b,\mathbf{s}_7^b)\ : an_2\sqrt{\frac{3T}{2}}-an_1\sqrt{\frac{3T}{2}}+\frac{3a^2T}{4}>-an_2\sqrt{3T}-\frac{3a^2T}{4}\left(\sqrt{2}+1\right)\ \to\frac{a}{2}\sqrt{3T}\left(\sqrt{2}+1\right)+n_2\left(\sqrt{2}+1\right)>n_1$$

$$:bn_2\sqrt{2}-bn_1\sqrt{2}+b^2>-2bn_2-b^2\left(\sqrt{2}+1\right)\ \to b\left(\sqrt{2}+1\right)+n_2\left(\sqrt{2}+1\right)>n_1$$

$$C(\mathbf{r}^b,\mathbf{s}_4^b)>C(\mathbf{r}^b,\mathbf{s}_8^b)\ : an_2\sqrt{\frac{3T}{2}}-an_1\sqrt{\frac{3T}{2}}+\frac{3a^2T}{4}>an_1\sqrt{\frac{3T}{2}}-an_2\sqrt{\frac{3T}{2}}-\frac{9a^2T}{4}\ \to a\sqrt{\frac{3T}{2}}+n_2>n_1$$

$$:bn_2\sqrt{2}-bn_1\sqrt{2}+b^2>bn_1\sqrt{2}-bn_2\sqrt{2}-3b^2\ \to b\sqrt{2}+n_2>n_1 \tag{10}$$

The above are useful in defining the correct decision regions, but need coordinate transformation for proper utilization, since the transmitted signal has components both from $\psi_1(t)$ and $\psi_2(t)$. Instead we appropriately rotate Constellation a and b right from the beginning, so that the signal vectors $\mathbf{s}_4^a$ and $\mathbf{s}_4^b$ are aligned along $\psi_1(t)$ axis. It is clear that such rotation will have no effect on the result. From (7) and (10), we see that it will be sufficient to consider the correct decision region the transmitted signal and the two adjacent signal vectors, in this case, these are $\mathbf{s}_3^a$, $\mathbf{s}_4^a$ and



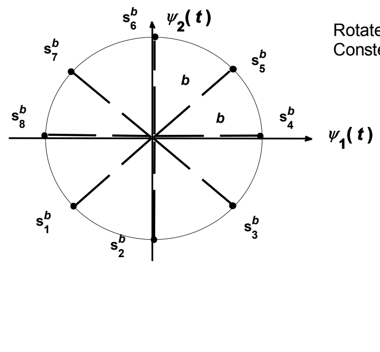


Fig. 5. Rotated Constellations a and b.

$\mathbf{s_5^a}$ in Constellation a, $\mathbf{s_3^b}$, $\mathbf{s_4^b}$ and $\mathbf{s_5^b}$ in Constellation b and the signal vectors at 180^0, that is $\mathbf{s_8^a}$ in Costellation a, $\mathbf{s_8^b}$ in Constellation b. Hence below we undertake the computation only for these signal vectors.

For the rotated constellations, the time waveforms, $s_3^a(t),\ s_4^a(t),\ s_5^a(t),\ s_8^a(t),\ s_3^b(t),\ s_4^b(t),\ s_5^b(t),\ s_8^b(t)$ and the signal vectors $\mathbf{s_3^a}$, $\mathbf{s_4^a}$, $\mathbf{s_5^a}$, $\mathbf{s_3^b}$, $\mathbf{s_4^b}$, $\mathbf{s_5^b}$ become

$$s_3^a(t)=\begin{cases}\dfrac{a}{2} & 0\le t\le T/2\\ -\dfrac{a}{2} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_3^a(t)=\frac{a}{2}\sqrt{T}\psi_1(t)-\frac{a}{2}\sqrt{T}\psi_2(t),\ \mathbf{s_3^a}=\left[s_{31}^a,\ s_{32}^a\right]=\left[\frac{a}{2}\sqrt{T},\ -\frac{a}{2}\sqrt{T}\right],\ \varepsilon_{s_3^a}=\frac{a^2T}{2}$$

$$s_4^a(t)=\begin{cases}a\sqrt{2} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases},\ s_1^a(t)=a\sqrt{T}\psi_1(t),\ \mathbf{s_4^a}=\left[s_{41}^a,\ s_{42}^a\right]=\left[a\sqrt{T},\ 0\right],\ \varepsilon_{s_4^a}=a^2T$$

$$s_5^a(t)=\begin{cases}\dfrac{a}{2} & 0\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_3^a(t)=\frac{a}{2}\sqrt{T}\psi_1(t)+\frac{a}{2}\sqrt{T}\psi_2(t),\ \mathbf{s_5^a}=\left[s_{51}^a,\ s_{52}^a\right]=\left[\frac{a}{2}\sqrt{T},\ \frac{a}{2}\sqrt{T}\right],\ \varepsilon_{s_5^a}=\frac{a^2T}{2}$$

$$s_8^a(t)=\begin{cases}-a\sqrt{2} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases},\ s_8^a(t)=-a\sqrt{T}\psi_1(t),\ \mathbf{s_8^a}=\left[s_{81}^a,\ s_{82}^a\right]=\left[-a\sqrt{T},\ 0\right],\ \varepsilon_{s_8^a}=a^2T$$

$$s_3^b(t)=\begin{cases}\dfrac{a}{2}\sqrt{3}=\dfrac{b}{\sqrt{T}} & 0\le t\le T/2\\ -\dfrac{a}{2}\sqrt{3}=\dfrac{-b}{\sqrt{T}} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_4^b(t)=\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_1(t)-\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_2(t)=\frac{b}{\sqrt{2}}\psi_1(t)-\frac{b}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_3^b}=\left[s_{31}^b,\ s_{32}^b\right]=\left[\frac{a}{2}\sqrt{\frac{3T}{2}},\ -\frac{a}{2}\sqrt{\frac{3T}{2}}\right]=\left[\frac{b}{\sqrt{2}},\ -\frac{b}{\sqrt{2}}\right],\ \varepsilon_{s_4^b}=\frac{3a^2T}{4}=b^2$$

$$s_4^b(t)=\begin{cases}a\sqrt{\dfrac{3}{2}}=b\sqrt{\dfrac{2}{T}} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases},\ s_4^b(t)=\frac{a}{2}\sqrt{3T}\psi_1(t)=b\psi_1(t)$$

$$\mathbf{s_4^b}=\left[s_{41}^b,\ s_{42}^b\right]=\left[\frac{a}{2}\sqrt{3T},\ 0\right]=\left[b,\ 0\right],\ \varepsilon_{s_4^b}=\frac{3a^2T}{4}=b^2$$

$$s_5^b(t)=\begin{cases}\dfrac{a}{2}\sqrt{3}=\dfrac{b}{\sqrt{T}} & 0\le t\le T/2\\ \dfrac{a}{2}\sqrt{3}=\dfrac{-b}{\sqrt{T}} & T/2\le t\le T\\ 0 & \text{otherwise}\end{cases},\ s_4^b(t)=\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_1(t)+\frac{a}{2}\sqrt{\frac{3T}{2}}\psi_2(t)=\frac{b}{\sqrt{2}}\psi_1(t)+\frac{b}{\sqrt{2}}\psi_2(t)$$

$$\mathbf{s_5^b}=\left[s_{51}^b,\ s_{52}^b\right]=\left[\frac{a}{2}\sqrt{\frac{3T}{2}},\ \frac{a}{2}\sqrt{\frac{3T}{2}}\right]=\left[\frac{b}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right],\ \varepsilon_{s_4^b}=\frac{3a^2T}{4}=b^2$$

$$s_8^b(t)=\begin{cases}-a\sqrt{\dfrac{3}{2}}=-b\sqrt{\dfrac{2}{T}} & 0\le t\le T/2\\ 0 & \text{otherwise}\end{cases}\ ,\ s_8^b(t)=-\frac{a}{2}\sqrt{3T}\psi_1(t)=-b\psi_1(t)$$

$$\mathbf{s_8^b}=\left[s_{81}^b,\ s_{82}^b\right]=\left[-\frac{a}{2}\sqrt{3T},\ 0\right]=\left[-b,\ 0\right]\ ,\ \varepsilon_{s_8^b}=\frac{3a^2T}{4}=b^2 \tag{11}$$

The received vectors computed from the output of correlator or matched filter are

$$\mathbf{r^a}=\begin{bmatrix}r_1^a\\ r_2^a\end{bmatrix}=\begin{bmatrix}a\sqrt{T}+n_1\\ n_2\end{bmatrix}\quad,\quad \mathbf{r^b}=\begin{bmatrix}r_1^b\\ r_2^b\end{bmatrix}=\begin{bmatrix}\dfrac{a}{2}\sqrt{3T}+n_1\\ n_2\end{bmatrix}=\begin{bmatrix}b+n_1\\ n_2\end{bmatrix} \tag{12}$$

As stated above, we evaluate correlation metrics $C\left(\mathbf{r^a},\ \mathbf{s_m^a}\right)$ and $C\left(\mathbf{r^b},\ \mathbf{s_m^b}\right)$ for $m=3,\ 4,\ 5,\ 8$

$$m=3\ ,\ C\left(\mathbf{r^a},\ \mathbf{s_3^a}\right)=2\,\mathbf{s_3^a}\cdot\mathbf{r^a}-\left\|\mathbf{s_3^a}\right\|^2=2\left[\frac{a}{2}\sqrt{T},\ -\frac{a}{2}\sqrt{T}\right]\begin{bmatrix}a\sqrt{T}+n_1\\ n_2\end{bmatrix}-a^2\frac{T}{2}=an_1\sqrt{T}-an_2\sqrt{T}+\frac{a^2T}{2}$$

$$m=4\ ,\ C\left(\mathbf{r^a},\ \mathbf{s_4^a}\right)=2\,\mathbf{s_4^a}\cdot\mathbf{r^a}-\left\|\mathbf{s_4^a}\right\|^2=2\left[a\sqrt{T},\ 0\right]\begin{bmatrix}a\sqrt{T}+n_1\\ n_2\end{bmatrix}-a^2T=2an_1\sqrt{T}+a^2T$$

$$m=5\ ,\ C\left(\mathbf{r^a},\ \mathbf{s_5^a}\right)=2\,\mathbf{s_5^a}\cdot\mathbf{r^a}-\left\|\mathbf{s_5^a}\right\|^2=2\left[\frac{a}{2}\sqrt{T},\ \frac{a}{2}\sqrt{T}\right]\begin{bmatrix}a\sqrt{T}+n_1\\ n_2\end{bmatrix}-a^2\frac{T}{2}=an_1\sqrt{T}+an_2\sqrt{T}+\frac{a^2T}{2}$$

$$m=8\ ,\ C\left(\mathbf{r^a},\ \mathbf{s_8^a}\right)=2\,\mathbf{s_8^a}\cdot\mathbf{r^a}-\left\|\mathbf{s_8^a}\right\|^2=2\left[-a\sqrt{T},\ 0\right]\begin{bmatrix}a\sqrt{T}+n_1\\ n_2\end{bmatrix}-a^2T=-2an_1\sqrt{T}-3a^2T$$

$$m=3\ ,\ C\left(\mathbf{r^b},\ \mathbf{s_3^b}\right)=2\,\mathbf{s_3^b}\cdot\mathbf{r^b}-\left\|\mathbf{s_3^b}\right\|^2=2\left[\frac{b}{\sqrt{2}},\ -\frac{b}{\sqrt{2}}\right]\begin{bmatrix}b+n_1\\ n_2\end{bmatrix}-b^2=bn_1\sqrt{2}-bn_2\sqrt{2}-b^2\left(1-\sqrt{2}\right)$$

$$m=4\ ,\ C\left(\mathbf{r^b},\ \mathbf{s_4^b}\right)=2\,\mathbf{s_4^b}\cdot\mathbf{r^b}-\left\|\mathbf{s_4^b}\right\|^2=2\left[b,\ 0\right]\begin{bmatrix}b+n_1\\ n_2\end{bmatrix}-b^2=2bn_1+b^2$$

$$m=5\ ,\ C\left(\mathbf{r^b},\ \mathbf{s_5^b}\right)=2\,\mathbf{s_5^b}\cdot\mathbf{r^b}-\left\|\mathbf{s_5^b}\right\|^2=2\left[\frac{b}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right]\begin{bmatrix}b+n_1\\ n_2\end{bmatrix}-b^2=bn_1\sqrt{2}+bn_2\sqrt{2}-b^2\left(1-\sqrt{2}\right)$$

$$m=8\ ,\ C\left(\mathbf{r^b},\ \mathbf{s_8^b}\right)=2\,\mathbf{s_8^b}\cdot\mathbf{r^b}-\left\|\mathbf{s_8^b}\right\|^2=2\left[-b,\ 0\right]\begin{bmatrix}b+n_1\\ n_2\end{bmatrix}-b^2=-2bn_1-3b^2 \tag{13}$$

To get the correct decision regions we must have $C\left(\mathbf{r^a},\ \mathbf{s_4^a}\right)>C\left(\mathbf{r^a},\ \mathbf{s_3^a}\right),\ C\left(\mathbf{r^a},\ \mathbf{s_4^a}\right)>C\left(\mathbf{r^a},\ \mathbf{s_5^a}\right),\ C\left(\mathbf{r^a},\ \mathbf{s_4^a}\right)>C\left(\mathbf{r^a},\ \mathbf{s_8^a}\right)$ and $C\left(\mathbf{r^b},\ \mathbf{s_4^b}\right)>C\left(\mathbf{r^b},\ \mathbf{s_3^b}\right),\ C\left(\mathbf{r^b},\ \mathbf{s_4^b}\right)>C\left(\mathbf{r^b},\ \mathbf{s_5^b}\right),\ C\left(\mathbf{r^b},\ \mathbf{s_4^b}\right)>C\left(\mathbf{r^b},\ \mathbf{s_8^b}\right)$. Hence, the followings

$$C\left(\mathbf{r}^{\mathrm{a}},\mathbf{s}_4^{\mathrm{a}}\right)>C\left(\mathbf{r}^{\mathrm{a}},\mathbf{s}_3^{\mathrm{a}}\right):2an_1\sqrt{T}+a^2T>an_1\sqrt{T}-an_2\sqrt{T}+\frac{a^2T}{2}\ \rightarrow\ \frac{a}{2}\sqrt{T}+n_1>-n_2$$

$$C\left(\mathbf{r}^{\mathrm{a}},\mathbf{s}_4^{\mathrm{a}}\right)>C\left(\mathbf{r}^{\mathrm{a}},\mathbf{s}_5^{\mathrm{a}}\right):2an_1\sqrt{T}+a^2T>an_1\sqrt{T}+an_2\sqrt{T}+\frac{a^2T}{2}\ \rightarrow\ \frac{a}{2}\sqrt{T}+n_1>n_2$$

$$C\left(\mathbf{r}^{\mathrm{a}},\mathbf{s}_4^{\mathrm{a}}\right)>C\left(\mathbf{r}^{\mathrm{a}},\mathbf{s}_8^{\mathrm{a}}\right):2an_1\sqrt{T}+a^2T>-2an_1\sqrt{T}-3a^2T\ \rightarrow\ a\sqrt{T}>-n_1$$

$$C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_4^{\mathrm{b}}\right)>C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_3^{\mathrm{b}}\right):2bn_1+b^2>bn_1\sqrt{2}-bn_2\sqrt{2}-b^2\left(1-\sqrt{2}\right)\ \rightarrow\ \left(\sqrt{2}-1\right)\left(b+n_1\right)>-n_2$$

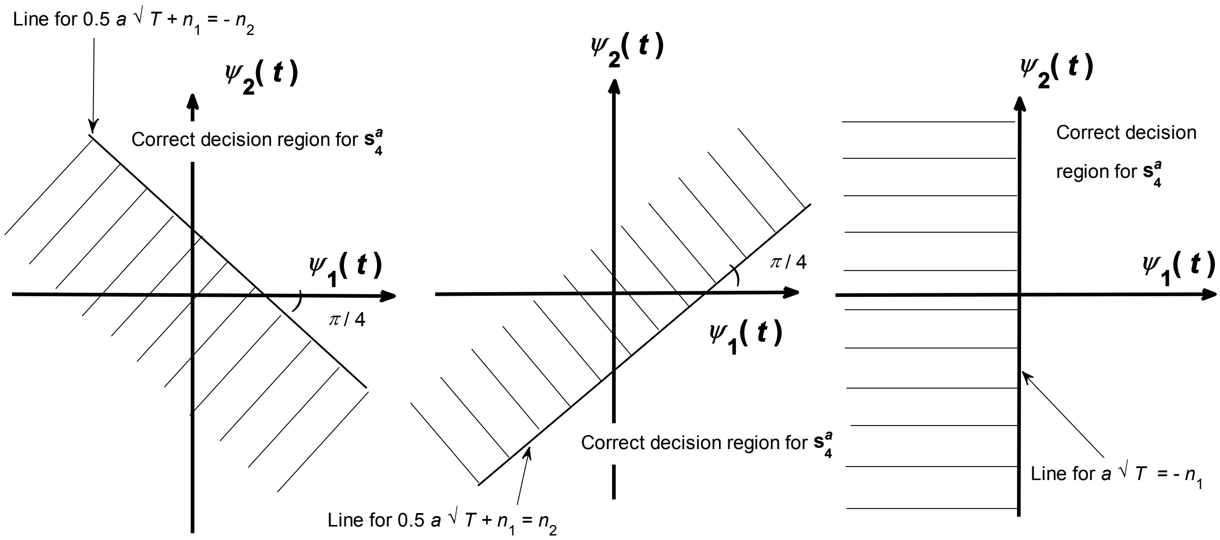
$$\rightarrow\ \left(\sqrt{2}-1\right)\left(\frac{a}{2}\sqrt{T}+\frac{n_1}{\sqrt{3}}\right)>-\frac{n_2}{\sqrt{3}}$$

$$C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_4^{\mathrm{b}}\right)>C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_5^{\mathrm{b}}\right):2bn_1+b^2>bn_1\sqrt{2}+bn_2\sqrt{2}-b^2\left(1-\sqrt{2}\right)\ \rightarrow\ \left(\sqrt{2}-1\right)\left(b+n_1\right)>n_2$$

$$\rightarrow\ \left(\sqrt{2}-1\right)\left(\frac{a}{2}\sqrt{T}+\frac{n_1}{\sqrt{3}}\right)>\frac{n_2}{\sqrt{3}}$$

$$C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_4^{\mathrm{b}}\right)>C\left(\mathbf{r}^{\mathrm{b}},\mathbf{s}_8^{\mathrm{b}}\right):2bn_1+b^2>-2bn_1-3b^2\ \rightarrow\ b>-n_1 \tag{14}$$

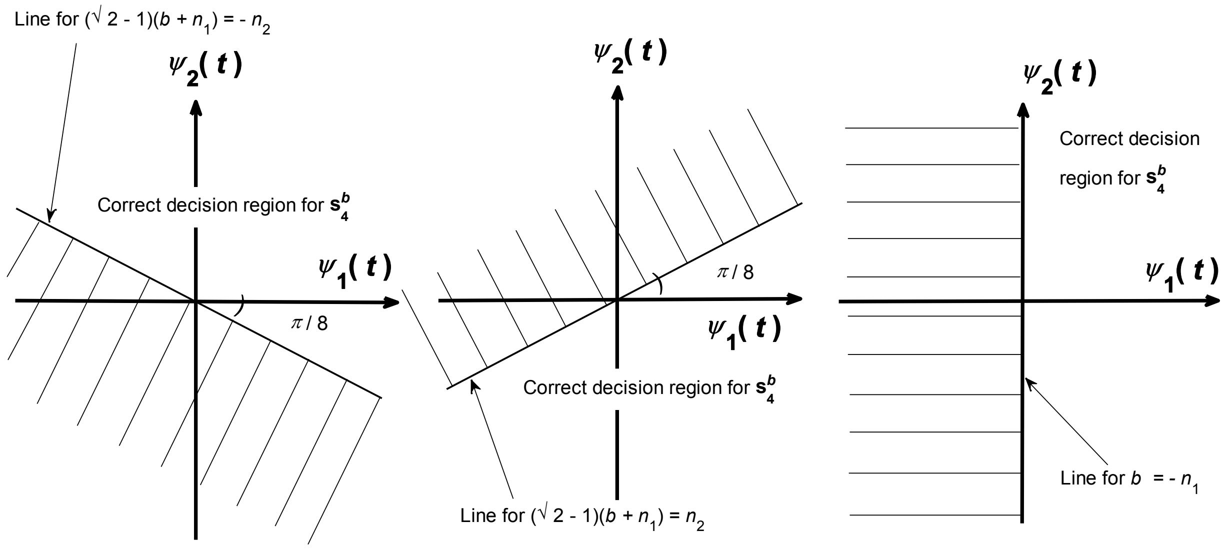
It is clear from the fourth, fifth and sixth, seventh lines of (14) that the correct decision regions between $\mathbf{s}_4^{\mathrm{b}}$ and $\mathbf{s}_3^{\mathrm{b}}$, $\mathbf{s}_5^{\mathrm{b}}$ are lines oriented at 22.5^0 to the horizontal $\psi_1(t)$ axis. From there if a comparison is made to first and second lines of (14), we see that the correct decision regions between $\mathbf{s}_4^{\mathrm{a}}$ and $\mathbf{s}_3^{\mathrm{a}}$, $\mathbf{s}_5^{\mathrm{a}}$ is a region with an angle larger than 22.5^0. This is due to the fact the length (or the energy) of $\mathbf{s}_4^{\mathrm{a}}$ is more than the lengths of $\mathbf{s}_3^{\mathrm{a}}$, $\mathbf{s}_5^{\mathrm{a}}$. In Fig. 6 and (15), we respectively show the regions defined by (14) and the probability of error.



a) Decision regions for $\mathbf{s}_4^{\mathrm{a}}$

Note that Fig. 6 could also have been plotted by labelling the axes as r_1 and r_2, where the transmitted signal will have a vectorial component along r_1 only. Thus $r_1=\left(\text{Energy of transmitted signal}\right)^{0.5}+n_1,\ r_2=n_2$. This way, the second line of (14) will become

$$\frac{a}{2}\sqrt{T}+n_1>n_2\rightarrow r_2<r_1-\frac{a}{2}\sqrt{T} \tag{15}$$



b) Decision regions for $\mathbf{s}_4^{\mathrm{b}}$

Fig. 6 Decision regions for $\mathbf{s}_4^{\mathrm{a}}$ and $\mathbf{s}_4^{\mathrm{b}}$.

For probability of error, we benefit from (8) and (9) of SampleProblems of Proakis2002_Nov 2012 and write the following

$$P_c = \frac{1}{(\pi N_0)^{0.5}} \int_{-a\sqrt{T}}^{\infty} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \frac{1}{(\pi N_0)^{0.5}} \int_{-\left(\frac{a}{2}\sqrt{T}+n_1\right)}^{\frac{a}{2}\sqrt{T}+n_1} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \qquad \text{for } \mathbf{s}_4^{\mathrm{a}}$$

$$P_c = \frac{1}{(\pi N_0)^{0.5}} \int_{-b}^{\infty} \exp\left(-\frac{n_1^2}{N_0}\right) dn_1 \frac{1}{(\pi N_0)^{0.5}} \int_{-(\sqrt{2}-1)(b+n_1)}^{(\sqrt{2}-1)(b+n_1)} \exp\left(-\frac{n_2^2}{N_0}\right) dn_2 \qquad \text{for } \mathbf{s}_4^{\mathrm{b}} \;,\; P_e = 1 - P_c \qquad (16)$$

As a check to the determination of our decision regions, we apply the simple rule that we define a point on the constellation diagram and then demand that the distance of this point to the transmitted signal vector end should be smaller than the distance to the adjacent signal vector end. This is a reasonable approach since this is the concept in finding the distance metric (the opposite of correlation metrics). If we take the signal vectors $\mathbf{s}_3^{\mathrm{a}}$ and $\mathbf{s}_4^{\mathrm{a}}$ from Fig. 5, rotated constellation a and name the arbitrary coordinates as r_1 (component along $\psi_1(t)$ axis) and r_2 (component along $\psi_2(t)$ axis), then making the distance from $[r_1,\, r_2]$ to $\mathbf{s}_4^{\mathrm{a}} = \left[a\sqrt{T},\, 0\right]$ smaller than the distance from $[r_1,\, r_2]$ to $\mathbf{s}_3^{\mathrm{a}} = \left[\frac{a}{2}\sqrt{T},\, -\frac{a}{2}\sqrt{T}\right]$, we get

$$\left(r_1 - a\sqrt{T}\right)^2 + r_2^{\,2} < \left(r_1 - \frac{a}{2}\sqrt{T}\right)^2 + \left(r_2 + \frac{a}{2}\sqrt{T}\right)^2$$

$$r_2 > -r_1 + \frac{a}{2}\sqrt{T} \text{ , set } r_1 = a\sqrt{T} + n_1 \text{ , } r_2 = n_2 \text{ then } n_2 > -n_1 - \frac{a}{2}\sqrt{T} \qquad (17)$$

Note the settings on the second line of (17). Further note that the last equality on the second line of (17) is identical to the last inequality of the first line of (14).

2. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer

a) ISI occurs because rectangular symbols are transmitted : Not exactly, ISI may also occur for types of signals not having the rectangular shape, where not all the transmitted signal energy will pass through the communication channel.

b) Matched filter takes the inverse of the input signal : False. Matched filter adopts itself to the input signal, so along time axis it has a time inverted and delayed response with respect to input.

c) In PSK, symbol duration increases as M increases : False, for any modulation system, if the binary waveform input is kept at fixed rate, then symbol duration will be reduced as M is increased.

d) As we increase the signal energy, the distances between signal vector ends increase : True, (provided that noise spectral density remains the same), when the signal is increased, the length of signal vectors increase, thus the distance between signal vector ends become greater.

e) To get the construction of an optimum detector, we consider the statistical properties of noise : True, the construction of an optimum detector is based on the consideration of noise pdf.

f) ASK is more efficient than PSK in terms of bandwidth used : True, ASK symbols extend over the entire duration of one symbol interval, whereas PSK symbols divide the symbol duration into two slices.